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Representations for the Drazin inverses of the sum of two matrices and some block matrices ☆

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ABSTRACT

In this paper, we give some formulas of the Drazin inverses of the sum of two matrices under the conditions $P^2Q = 0,\ Q^2P = 0$ and $P^3Q = 0,\ QPQ = 0,\ QP^2Q = 0$ respectively. And we also give some representations for the Drazin inverse of block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ (A and D are square) under some conditions.

1. Introduction

Let $\mathbb{C}^{m\times n}$ denote the set of $m \times n$ complex matrices. For $A \in \mathbb{C}^{n\times n}$, we call the smallest nonnegative integer k which satisfies $rank(A^{k+1}) = rank(A^k)$ the index of A, and denote k by $ind(A)$. Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$, we call the matrix $X \in \mathbb{C}^{n\times n}$ which satisfies

$$A^{k+1}X = A^k, \quad XAX = X, \quad AX = XA,$$

the Drazin inverse of A and denote X by A^D (see [1]). The Drazin inverse of a square matrix exists and is unique (see [1]). If $ind(A) = 1$, then we call A^D the group inverse of A and denote it by $A^{\#}$. In this paper, we denote $A^{\pi} = I - AA^D$.

The Drazin inverse of a square matrix is widely applied in singular differential or difference equations, iterative method and perturbation bounds for the relative eigenvalue problem (see [1–4]), respectively. The Drazin inverse was shown to be a central object in Markov chain theory for finite state chains by Campbell and Meyer (see [1]), and by Spitzner and Boucher for general state chains (see [5]).

In 1977, Meyer and Rose gave the explicit representation of the Drazin inverse for block matrix $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ with A and C are square (see [6]). In 1979, Campbell and Meyer proposed an open problem to find the explicit representation of the Drazin inverse for block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with A and D are square. In 1983, Campbell proposed a problem to find the representation of $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^D$ (A is a singular square matrix) on the basis of finding the solution of systems of second order linear differential equation such as $Ex''(t) + Fx'(t) + Gx(t) = 0$, where E is singular (see [2]). These problems have not been solved till now.

In 1958, Drazin gave the representation of $(P+Q)^D$ with P and Q are square matrices firstly (see [7]). Herein, it was proved that

$$(P+Q)^D = P^D + Q^D \quad \text{when } QP = PQ = 0.$$

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In 2001, Hartwig et al. gave the representation of $(P+Q)^D$ when $PQ=0$ (see [3]). In 2009, Martínez-Serrano and Castro-González gave the representation of $(P+Q)^D$ when $P^2Q=0$ and $Q^2=0$ (see [8]). In 2011, Yang and Liu gave the representation of $(P+Q)^D$ when $P^2Q=0$ and $QPQ=0$ (see [9]). The results about the representation of $(P+Q)^D$ are useful in computing the representations of the Drazin inverse for block matrices, analyzing a class of perturbation and iteration theory. The general questions of how to express $(P+Q)^D$ by P,Q,P^D,Q^D and $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^D$ by A,B,C,D without side condition are very difficult and have not been solved. However, some results about the representations of $(P+Q)^D$ and $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^D$ under some conditions were given. Here we list the results below:

(1) Results of the representations of $(P+Q)^D$ under the following conditions respectively:
(1-1) $PQ=0$ (see [3]);
(1-2) $P^DQ=0,\ PQ^D=0,\ Q^\pi PQP^\pi=0$ (see [10]);
(1-3) $P^2Q+PQ^2=0$ (see [11]);
(1-4) $P^2Q=0,\ Q^2=0$ (see [8]);
(1-5) $PQ=QP$ (see [12]);
(1-6) $P^3Q=QP,\ Q^3P=PQ$ (see [13]);
(1-7) $P^2Q=0,\ QPQ=0$ (see [9]).
(2) Results of the representations of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^D$ under the following conditions respectively:
(2-1) $CA^\pi=0,\ A^\pi B=0$, generalized Schur complement $D-CA^DB=0$ (see [14]);
(2-2) $CA^\pi=0,\ A^\pi B=0$, generalized Schur complement $D-CA^DB$ is nonsingular (see [15]);
(2-3) $A=I,\ DC=0$ (see [16]);
(2-4) $BC=0,\ DC=0$ or $BD=0$, D is nilpotent (see [17]);
(2-5) $A^2A^\pi B=0,\ CAA^\pi B=0,\ BCA^\pi B=0$, generalized Schur complement $D-CA^DB=0$ (see [8]);
(2-6) $BD^iC=0,\ i=0,1,\dots,n-1$ (see [18]);
(2-7) $BCA^\pi=0,\ D=0,\ CA^DB$ is nonsingular (see [19]);
(2-8) $BD^\pi C=0,\ BDD^D=0,\ DD^\pi C=0$ (see [20]);
(2-9) $C=I,\ D=0,\ AB=BA$ (see [21]);
(2-10) $A=B=A^2,\ D=0$ (see [22]);
(2-11) $ABC=0,\ D=0$ (see [23]).

In this paper, we first give the formulas of $(P+Q)^D$ under the conditions $P^2Q=0,\ Q^2P=0$ and $P^3Q=0,\ QPQ=0,\ QP^2Q=0$ respectively. These results extend the formula (1-4) and (1-7) above respectively. And then we use the formulas of $(P+Q)^D$ to give some representations for the Drazin inverse of block matrix $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ (A and D are square) under some conditions:

(i) $ABCA^\pi=0,\ A^\pi ABC=0$ and $S=D-CA^DB$ is zero (it generalizes the Theorem 3.6 in [8]);
(ii) $D=0,\ CA^DBC=0,\ ABCA^\pi=0,\ A^\pi ABC=0$ (it generalizes the Theorem 3.3 in [24]);
(iii) $CBCA^\pi=0,\ ABCA^\pi=0$ and $S=D-CA^DB$ is zero (it generalizes the Corollary 3.5 in [8]);
(iv) $CA^\pi BC=0,\ A^2A^\pi BC=0,\ CAA^\pi BC=0$ and $S=D-CA^DB$ is zero (it generalizes the Theorem 3.3 in [9]).

2. Some lemmas

Before we give the main results, we first give some lemmas here.

Lemma 2.1 (see [25]). *Let $A\in\mathbb{C}^{m\times n},\ B\in\mathbb{C}^{n\times m}$, then $(AB)^D=A((BA)^D)^2B$.*

Lemma 2.2 (see [6]). *Let $M=\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\in\mathbb{C}^{n\times n}$ (A and C are square), $k=ind(A)$ and $j=ind(C)$, then*

$$M^D=\begin{pmatrix} A^D & X \\ 0 & C^D \end{pmatrix},$$

where $X=\sum_{i=0}^{j-1}(A^D)^{i+2}BC^iC^\pi+A^\pi\sum_{i=0}^{k-1}A^iB(C^D)^{i+2}-A^DBC^D$.

Lemma 2.3 (*see [3]*). *Let $P,Q\in\mathbb{C}^{n\times n}$, if $PQ=0$, then*

$$(P+Q)^D=Q^\pi\sum_{i=0}^{t-1}Q^i(P^D)^{i+1}+\sum_{i=0}^{t-1}(Q^D)^{i+1}P^iP^\pi,$$

where $t=\max\{ind(P),ind(Q)\}$.