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Asymptotic solutions for singularly perturbed Boussinesq equations

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ABSTRACT

We consider a family of singularly perturbed Boussinesq equations. We obtain a rational weak solution to the classical Boussinesq equation and demonstrate that this solution can be used to construct perturbation solutions for singularly perturbed high-order Boussinesq equations. These solutions take the form of an algebraic function which behaves similarly to a peakon, and which decays as time becomes large. We show that approximate solutions obtained via perturbation for the singularly perturbed models are asymptotic to the true solutions as the residual errors rapidly decay away from the origin.

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1. Introduction

The Boussinesq equation, which describes bi-directional surface wave propagation, was formulated by Boussinesq [1]:

$$u_{tt} - u_{xx} - u_{xxxx} + (u^2)_{xx} = 0. \tag{1}$$

The basic physical properties of this equation are discussed in Boussinesq [1], and exact travelling wave solutions of exponential, trigonometric, and algebraic types are given in Bruzon [2]. Multiple soliton solutions are given in Wazwaz [3], including solutions of trigonometric and hyperbolic trigonometric types. In Ma et al. [4], a Wronskian formulation leads to rational solutions to a modified version of the equation. In Feng [5], travelling solitary wave solutions to a far more general Boussinesq equation are obtained. Chavez et al. [6] give numerical solutions and an application to agricultural engineering, and Sun et al. [7] give an approximate solution and an application to hydrologic engineering.

In some settings the classical Boussinesq equation is ill-posed: Turitsyn [8] discusses the collapse dynamics arising with periodic boundary conditions, and a sufficient condition for the initial value problem to develop a singularity. Difficulties in dealing with the equation numerically are discussed by Daripa and Hua [9], where they formulate a perturbed Boussinesq equation with a higher order term:

$$u_{tt} - u_{xx} - u_{xxxx} + (u^2)_{xx} = \epsilon u_{xxxxxx}. \tag{2}$$

Dash and Daripa [10] show that this perturbed Boussinesq equation describes the bi-directional propagation of certain small amplitude and long capillary-gravity waves, and provide analysis of far-field behaviour. Jiao et al. [11] study the perturbed equation with a homotopy analysis method and find approximate series solutions which have low error near the origin.

In this brief paper, we consider a family of singularly perturbed Boussinesq equations. We obtain a rational weak solution to the classical Boussinesq equation and demonstrate that this solution can be used to construct perturbation solutions for singularly perturbed high-order Boussinesq equations. We show that the obtained solutions are asymptotic to the true solutions as the residual errors rapidly decay away from the origin.

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2. Exact rational solution to the Boussinesq equation

Consider the Boussinesq equation (1). We see that the function

$$u(x,t) = \frac{6}{(x+t+r)^2} \tag{3}$$

satisfies the Boussinesq equation (1) for any real number r. We notice that this function has uncountably many singularities. To deal with this, we require that $r > 0$ and modify the formula slightly:

$$u(x,t) = \frac{6}{(|x|+|t|+r)^2}. \tag{4}$$

Notice that this function has no singularities, but is not differentiable at $x = 0$ or $t = 0$. However, all the directional derivatives do exist at those points, and this function does indeed satisfy the Boussinesq equation. Since the function is not differentiable at $x = 0$, (4) is a weak solution to the Boussinesq equation. As such, for each $r > 0$ we have a weak solution to the Boussinesq equation. Note that this solution satisfies the initial condition

$$u(x,0) = \frac{6}{(\mid x \mid + r)^2}, \tag{5}$$

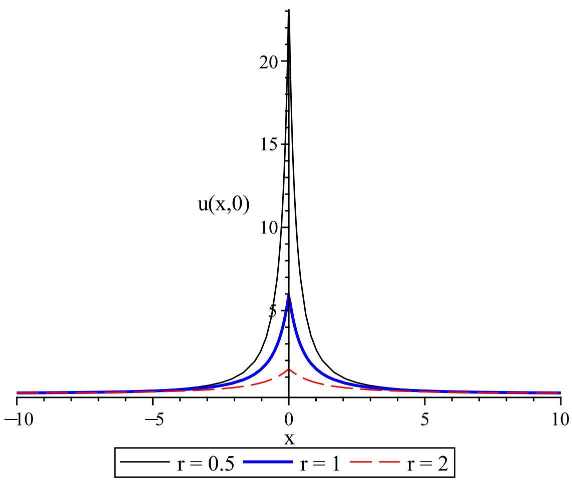

Fig. 1. Peaked initial data profiles $u(x,0)$ for the classical Boussinesq equation permitting rational solutions. Various values of $r > 0$ are chosen in order to demonstrate the dependence of initial data on various values of r.

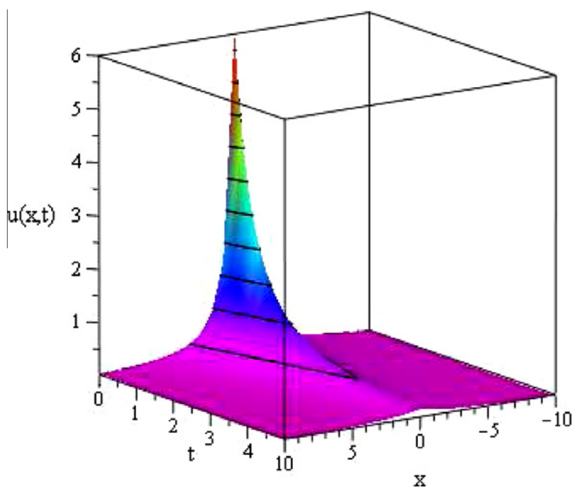

Fig. 2. Rational weak solutions $u(x,t)$ for the classical Boussinesq equation. Here we fix the parameter $r = 1$.