



## Comments on ‘A finite extensibility nonlinear oscillator’

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### ABSTRACT

The aim of this comment is to provide more information about the study of the dynamics of a finite extensibility nonlinear oscillator conducted by Febbo [M. Febbo, A finite extensibility nonlinear oscillator, *Applied Mathematics and Computation* 217 (2011) 6464–6475]. We show that the linearized harmonic balance method is not sufficiently adequate for this oscillator and that the harmonic balance method (HBM) without linearization provides better results. We also discuss what happens when the oscillation amplitude approaches 1 and why the harmonic balance method does not give optimum results. For these values of the oscillation amplitude the periodic solution becomes markedly anharmonic and is almost straight between  $x = +A$  and  $x = -A$  (with negative slope) and between  $x = -A$  and  $x = +A$  (with positive slope). Finally, a ‘heuristic’ solution is proposed which is adequate for the whole amplitude range  $0 < A < 1$ , which is consistent with the approximate solution obtained previously for  $A < 0.9$  using the HBM.

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## 1. Introduction

In a recent paper [1], Febbo studied analytically the dynamics of a finite extensibility nonlinear oscillator (FENO) using two different approaches. One involved a linearized harmonic balance (LHB) procedure, which allowed him to obtain analytical approximations to the frequency of oscillations and periodic solution. In Febbo's paper the approximate period obtained using a LHB method is compared with the exact one (numerically integrated) and very good agreement is obtained for amplitudes ( $A$ ) between 0 and 0.9 with a relative error of less than 3.53%. However, for the rest of the amplitude range ( $0.9 < A < 1$ ), the relative error for the approximate period increases exponentially and the author mentioned that higher order perturbation solutions are needed in such cases.

In this note, we would first like to take the opportunity to congratulate the author of Ref. [1] for his interesting, comprehensive study of a finite extensibility nonlinear oscillator. We will then add some interesting results about application of the harmonic balance method to a finite extensibility nonlinear oscillator – whether or not it is better to use a linearized version of this procedure – which are not included in Ref. [1] and compare the approximate and exact periodic solutions. Our results may provide information about why approximate methods fail when the oscillation amplitude approaches 1. In particular, we show that linearized harmonic balance procedures, even though they simplify the harmonic balancing, do not always provide optimal results. For instance, if the second order harmonic balance method without linearization is applied to a finite extensibility nonlinear oscillator, the relative error is as low as 0.60% for  $A = 0.9$ , whereas this error is 3.53% when a linearized

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harmonic balance method is used. This means that great care must be exercised when linearized harmonic balance procedures are applied to nonlinear oscillators with very strong nonlinearities. Finally, we present some comments on the behavior of the periodic solutions for amplitudes approaching 1, which may not only enable us to understand this type of nonlinear oscillator better, but also provide the basis for a more detailed study of the dynamics of a finite extensibility nonlinear oscillator and an extension of Febbo's paper.

## 2. Application of the harmonic balance method

The non-dimensional equation of motion governing a finite extensibility nonlinear oscillator is [1]

$$\frac{d^2x}{dt^2} + \frac{x}{1-x^2} = 0 \quad (1)$$

with initial conditions

$$x(0) = A \quad (\text{with } 0 < A < 1) \quad \text{and} \quad \frac{dx}{dt}(0) = 0, \quad (2)$$

where  $A$  is the oscillation amplitude.

The harmonic balance (HB) method provides a technique for calculating analytical approximations to the periodic solutions of differential equations by using a truncated Fourier series [2–4]. As we mentioned in the introduction, the HB method without linearization will be applied to a finite extensibility nonlinear oscillator.

Before applying the HB method to Eq. (1), this equation is rewritten in a form that does not contain the fractional expression

$$(1-x^2) \frac{d^2x}{dt^2} + x = 0. \quad (3)$$

Introducing a new independent variable  $\tau = \omega t$ , where  $\omega$  is the frequency of the oscillations, Eqs. (2) and (3) can be rewritten as

$$\omega^2(1-x^2) \frac{d^2x}{d\tau^2} + x = 0, \quad x(0) = A \quad (0 < A < 1), \quad \frac{dx}{d\tau}(0) = 0. \quad (4)$$

The new independent variable  $\tau$  is chosen in such a way that the solution of Eq. (4) is a periodic function of  $\tau$  of period  $2\pi$  [3]. Applying the lowest harmonic balance method it is easy to obtain the following first-order analytical approximate frequency (Eq. (32) in Febbo's paper [1])

$$\omega_1(A) = \frac{2}{\sqrt{4-3A^2}} \quad (5)$$

and the corresponding approximate periodic solution is  $x_1(t) = A \cos \omega_1 t$ . The second-order approximate solution to Eq. (4) can be expressed as

$$x_2(\tau) = A \cos \tau + c_1(\cos 3\tau - \cos \tau), \quad (6)$$

which satisfies the initial conditions in Eq. (4) and where  $c_1$  depends on the initial amplitude  $A$ . In Febbo's paper [1], Eq. (6) is substituted in Eq. (4) and higher-order corrections in  $c_1$  are discarded. This is the linearized harmonic balance (LHB) method that Febbo mentioned. This approximation is usually sufficient due to the low values of  $c_1$ . However, as we can see in this note, when  $A$  approaches 1, the higher-order corrections in  $c_1$  are important and can not be discarded. Then, the LHB method does not give accurate approximations to the frequency and the periodic solution when  $A$  increases (for  $A = 0.9$ , Febbo obtained a relative error of 3.53% for the approximate period using the LHB method, see Table 1 in [1]).

Substituting Eq. (6) into Eq. (4), expanding the expression in a trigonometric series, simplifying and setting the coefficient of the resulting items  $\cos \tau$  and  $\cos 3\tau$  equal to zero yields

$$(4-3A^2-5c_1A-30c_1^2)\omega^2-4=0 \quad (7)$$

and

$$[A^3-(36-19A^2)c_1-41c_1^2A+48c_1^3]\omega^2+4c_1=0. \quad (8)$$

From Eq. (7) the second approximate frequency  $\omega_2$  can be obtained as follows

$$\omega_2(A) = \frac{2}{\sqrt{4-3A^2-5Ac_1(A)-30c_1^2(A)}}. \quad (9)$$

Substituting Eq. (9) into Eq. (7) and simplifying, the following cubic equation is obtained

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