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Local existence of Lipschitz-continuous solutions of systems of nonlinear functional equations with iterated deviations

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ABSTRACT

Some conditions which guarantee the local existence of Lipschitz-continuous solutions of some systems of nonlinear functional equations with complicated deviations dependent of unknown functions are presented.

1. Introduction

Studying nonlinear discrete difference equations or systems attracted considerable attention recently, see, e.g., [4–6,8,9,13,15–17,26–36,38] and the related references therein, while nonlinear difference equations with continuous argument nowadays are not of such interest. On the other hand, constant interest exists in studying nonlinear functional equations with continuous argument.

Functional equations of the next type

$$x(t) = f(t, x(\alpha_1 t + \varphi_1(t, x(t))), \dots, x(\alpha_k t + \varphi_k(t, x(t)))), \tag{1}$$

where $\alpha_i \in \mathbb{R}$, $i = \overline{1,k}$, and $f(t, x_1, \dots, x_k)$ and $\varphi_i(t,x)$, $i = \overline{1,k}$, are real functions, have been considerably studied so far. One of the basic problems studied in the research area are the existence and uniqueness of specific type of solutions of certain classes of equations of this type (see, e.g., [1,2,10–12,14,18–20,39]). For related equations and results see also papers [3,7] and the references therein.

Some iteration methods for approximating fixed points which are sorts of iterations of some iterative processes, were introduced in our papers [21–25]. This motivated us to study functional equations with continuous arguments, whose deviations of an argument depend on an unknown function which depend also of the same function.

While in [37] we studied unique existence of bounded continuous solutions on the whole real line $\mathbb{R}$ of a class of nonlinear functional equations by a fixed point theorem approach, motivated by [19], here we will study the unique existence of local Lipschitz-continuous solutions of the following system of nonlinear functional equations on $\mathbb{R}$ by another approach

$$x(t) = f(t, x(\alpha_1 t + \varphi_1(t, x(\beta_1 t + \psi_1(t, x(t))))), \dots, x(\alpha_k t + \varphi_k(t, x(\beta_k t + \psi_k(t, x(t)))))), \tag{2}$$

where $\alpha_i,\ \beta_i \in \mathbb{R}, i = \overline{1,k},\ \varphi_i(t,x),\ \psi_i(t,x),\ i = \overline{1,k}$, are real functions, $f(t, x_1, \dots, x_k)$ is a real vector function, and $x(t)$ is an unknown vector function on a subset of $\mathbb{R}^n$.

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2. Main results

In this section we formulate and prove our main result in this paper which gives some sufficient conditions for the unique existence of local Lipschitz-continuous solutions of system (2).

Theorem 1. *Suppose $k \in \mathbb{N}$, $a, b > 0$, and that the following conditions hold:*

(i) *the function $f(t, x_1, \ldots, x_k)$ is continuous in the domain*

$$\mathcal{D} = \left\{ (t, x_1, \ldots, x_k) : |t| \leqslant a, \ |x_i| = \max_{1 \leqslant j \leqslant n} |x_i^{(j)}| \leqslant b, \ i = \overline{1, k} \right\},$$

the functions $\varphi_i(t, x)$, $\psi_i(t, x)$, $i = \overline{1, k}$, are continuous on the domain

$$\mathcal{D}_1 = \left\{ (t, x) : |t| \leqslant a, \ |x| = \max_{1 \leqslant j \leqslant n} |x^{(j)}| \leqslant b \right\}$$

and

$$f(0, 0, \ldots, 0) = 0, \quad \varphi_i(0, 0) = 0, \quad \psi_i(0, 0) = 0, \quad i = \overline{1, k}; \tag{3}$$

(ii) *the functions $f(t, x_1, \ldots, x_k)$, $\varphi_i(t, x)$ and $\psi_i(t, x)$, $i = \overline{1, k}$, satisfy the following Lipschitz conditions:*

$$|f(t_1, x_1, \ldots, x_k) - f(t_2, y_1, \ldots, y_k)| \leqslant L_0 |t_1 - t_2| + \sum_{j=1}^{k} L_j |x_j - y_j|, \tag{4}$$

$$|\varphi_i(t_1, x_1) - \varphi_i(t_2, x_2)| \leqslant l_i^{(1)} |t_1 - t_2| + l_i^{(2)} |x_1 - x_2|, \quad i = \overline{1, k}, \tag{5}$$

$$|\psi_i(t_1, x_1) - \psi_i(t_2, x_2)| \leqslant l_i^{(3)} |t_1 - t_2| + l_i^{(4)} |x_1 - x_2|, \quad i = \overline{1, k}, \tag{6}$$

where $L_0, L_i, l_i^{(j)}$, $j \in \{1, 2, 3, 4\}$, $i = \overline{1, k}$, are positive constants, $(t_1, x_1, \ldots, x_k), (t_1, y_1, \ldots, y_k) \in \mathcal{D}$, and $(t_1, x_1), (t_2, x_2) \in \mathcal{D}_1$;

(iii) *α_i, $\beta_i \in (0, 1)$, $i = \overline{1, k}$, and $M \in (L_0, \infty)$ is a constant such that*

$$\alpha_i + l_i^{(1)} + l_i^{(2)} M \left(\beta_i + l_i^{(3)} + l_i^{(4)} M \right) \leqslant 1, \quad i = \overline{1, k}, \tag{7}$$

$$\theta := M^2 \sum_{j=1}^{k} L_j l_j^{(2)} l_j^{(4)} + \sum_{j=1}^{k} L_j \left(\alpha_j + l_j^{(1)} + 2 l_j^{(2)} M \left(\beta_j + l_j^{(3)} + l_j^{(4)} M \right) \right) < 1, \tag{8}$$

$$L_0 + M \sum_{j=1}^{k} L_j \left(\alpha_j + l_j^{(1)} + l_j^{(2)} M \left(\beta_j + l_j^{(3)} + M l_j^{(4)} \right) \right) \leqslant M, \tag{9}$$

$$\beta_i + l_i^{(3)} + M l_i^{(4)} \leqslant 1, \quad i = \overline{1, k}. \tag{10}$$

Then, for sufficiently small $l_i^{(j)}$, $j \in \{1, 2, 3, 4\}$, $i = \overline{1, k}$, and for $|t| \leqslant a_*$, where $a_* = \min\{b/M, a\}$, there is a unique continuous solution $x(t)$ of system of equations (2) satisfying the Lipschitz condition

$$|x(t) - x(s)| \leqslant M |t - s| \tag{11}$$

for every t and s such that $|t| \leqslant a_*$ and $|s| \leqslant a_*$, and such that $x(0) = 0$.

Proof. We show the existence of the solution by the method of successive approximations. Let the sequence of vector functions $x_m(t)$, $m \in \mathbb{N}_0$, be defined by

$$x_0(t) = 0, \tag{12}$$

$$x_m(t) = f(t, x_{m-1}(\alpha_1 t + \varphi_1(t, x_{m-1}(\beta_1 t + \psi_1(t, x_{m-1}(t))))), \ldots, x_{m-1}(\alpha_k t + \varphi_k(t, x_{m-1}(\beta_k t + \psi_k(t, x_{m-1}(t)))))), \quad m \in \mathbb{N}. \tag{13}$$

Assume we have proved that $x_{m_0}(0) = 0$ for some $m \in \mathbb{N}$. From this, by (13) and using conditions (3), we have that

$$\begin{aligned} x_{m_0+1}(0) &= f(0, x_{m_0}(\varphi_1(0, x_{m_0}(\psi_1(0, x_{m_0}(0))))), \ldots, x_{m_0}(\varphi_k(0, x_{m_0}(\psi_k(0, x_{m_0}(0)))))) \\ &= f(0, x_{m_0}(\varphi_1(0, x_{m_0}(\psi_1(0, 0)))), \ldots, x_{m_0}(\varphi_k(0, x_{m_0}(\psi_k(0, 0))))) = f(0, x_{m_0}(\varphi_1(0, x_{m_0}(0))), \ldots, x_{m_0}(\varphi_k(0, x_{m_0}(0)))) \\ &= f(0, x_{m_0}(\varphi_1(0, 0)), \ldots, x_{m_0}(\varphi_k(0, 0))) = f(0, x_{m_0}(0), \ldots, x_{m_0}(0)) = f(0, 0, \ldots, 0) = 0. \end{aligned}$$

From this, (12), and by the method of induction we have that $x_m(0) = 0$ for every $m \in \mathbb{N}_0$.