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Gaps between zeros of second-order half-linear differential equations

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ABSTRACT

In this paper, for second order half-linear differential equations, we will establish some new inequalities of Lyapunov's type. These inequalities give results related to the spacing between consecutive zeros of a solution and the spacing between a zero of a solution and/or a zero of its derivative. The results also yield conditions for disfocality, disconjugacy and lower bounds for an eigenvalue of a boundary value problem. The main results will be proved by making use of some generalizations of Opial and Wirtinger type inequalities. Some examples are considered to illustrate the main results.

1. Introduction

Consider the second-order half-linear differential equation

$$\left(r(t)\mid x'(t)|^{\gamma-1}x'(t)\right)'+q(t)\mid x(t)|^{\gamma-1}x(t)=0,\}\quad \alpha\leqslant t\leqslant\beta, \tag{1.1}$$

where $\gamma\geqslant 1$ is a positive constant, r and q are real measurable functions on $[\alpha,\beta]$ satisfying $r(t)>0$, and

$$\int_\alpha^\beta\left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}ds<\infty\quad\text{and}\quad\int_\alpha^\beta\mid q(t)\mid dt<\infty.$$

The terminology half-linear arises because of the fact that the space of all solutions of (1.1) is homogeneous, but not generally additive. We assume that (1.1) possesses such a nontrivial solution. The nontrivial solution x of (1.1) is said to be oscillate or to be oscillatory, if it has arbitrarily large zeros. Eq. (1.1) is said to be disconjugate on the interval $[\alpha,\beta]$, if there is no nontrivial solution of (1.1) with two zeros in $[\alpha,\beta]$. Eq. (1.1) is said to be nonoscillatory on $[t_0,\infty)$ if there exists $c\in[t_0,\infty)$ such that this equation is disconjugate on $[c,d]$ for every $d>c$. We say that (1.1) is right disfocal (left disfocal) on $[\alpha,\beta]$ if the solutions of (1.1) such that $x'(\alpha)=0$ $(x'(\beta)=0)$ have no zeros in $[\alpha,\beta]$. For oscillation and nonoscillation results for half-linear differential equations, we refer the reader to the book [22].

In this paper, we are concerned with three interested problems for a solution $x(t)$ of the Eq. (1.1):

(i) obtain lower bounds for the spacing $\beta-\alpha$, where $x(\alpha)=x'(\beta)=0$, or $x'(\alpha)=x(\beta)=0$,
(ii) obtain lower bounds for the spacing $\beta-\alpha$, where $x(\alpha)=x(\beta)=0$,
(iii) obtain a lower bound for the first eigenvalue of the boundary value problem

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$$-\left((x'(t))^{\gamma}\right)' + q(t)x^{\gamma}(t) = \lambda x^{\gamma}(t), \quad x(\alpha) = x(\beta) = 0.$$

The best known existence result in the literature for a special case of Eq. (1.1) was proved by Lyapunov [16]. This result supplies a criterion sufficient for the stability of the differential equation

$$x'' + q(t)x(t) = 0, \quad \alpha \leqslant t \leqslant \beta, \tag{1.2}$$

with a periodic coefficient. This result together with [25] establishes that if $q(t)$ is positive continuous and if (1.2) has two zeros $a < b$, then

$$\int_a^b \mid q(t) \mid dt > \frac{4}{b-a}. \tag{1.3}$$

This result has been used in many applications, for example in eigenvalue problems, stability, etc. [28]. Since the discovery of this inequality many generalizations and extensions have appeared in the literature. In the following, we present some results that motivate the contents of this paper. In [15] the author shows that if x is a solution of (1.2) such that $x'(0) = x(b) = 0$, then

$$\int_0^b (b-t)q_+(t)dt > 1 \quad \text{where} \quad q_+ = \max\{q(t), 0\}, \tag{1.4}$$

and if

$$\int_a^c (t-a)q_+(t)dt \leqslant 1 \quad \text{and} \quad \int_c^b (b-t)q_+(t)dt \leqslant 1, \tag{1.5}$$

then (1.2) is disconjugate in $[a, b]$. In [12] the authors proved that if x is a solution of (1.2) with no zeros in (α, β) and such that $x'(\alpha) = x(\beta) = 0$, then

$$(\beta - \alpha) \sup_{\alpha \leqslant t \leqslant \beta} \left| \int_\alpha^t q(s)ds \right| > 1. \tag{1.6}$$

If instead $x(\alpha) = x'(\beta) = 0$, then

$$(\beta - \alpha) \sup_{\alpha \leqslant t \leqslant \beta} \left| \int_t^\beta q(s)ds \right| > 1. \tag{1.7}$$

Brown and Hinton [7] proved that if x is a solution of the Eq. (1.2) with no zeros in (α, β) and such that $x(\alpha) = x'(\beta) = 0$, then

$$2\int_\alpha^\beta Q^2(s)(s-\alpha)ds > 1, \tag{1.8}$$

where $Q(t) = \int_t^\beta q(s)ds$. If instead $x'(\alpha) = x(\beta) = 0$, then

$$2\int_\alpha^\beta Q^2(s)(\beta - s)ds > 1, \tag{1.9}$$

where $Q(t) = \int_\alpha^t q(s)ds$. In [2] the nonlinear differential equation

$$(r(t)x'(t))' + q(t)f(x(t)) = 0 \tag{1.10}$$

was considered and some new Lyapunov type inequalities were presented. One can find other results for (1.2) and higher order differential equations in [9–11,17,23,24]. Hong et al. [13] considered the equation

$$\left(\mid x'(t)\mid^{\gamma-1}x'(t)\right)' + q(t) \mid x(t)\mid^{\gamma-1}x(t) = 0, \quad \alpha \leqslant t \leqslant \beta, \tag{1.11}$$

where $\gamma \geqslant 1$ is a positive constant and proved that if x is a solution of the Eq. (1.11) with no zeros in (α, β) and such that $x'(\alpha) = x(\beta) = 0$, then

$$(\beta - \alpha)^{\gamma} \sup_{\alpha \leqslant t \leqslant \beta} \left| \int_\alpha^t q(s)ds \right| > 1. \tag{1.12}$$

If instead $x(\alpha) = x'(\beta) = 0$, then

$$(\beta - \alpha)^{\gamma} \sup_{\alpha \leqslant t \leqslant \beta} \left| \int_t^\beta q(s)ds \right| > 1. \tag{1.13}$$

In [20] the general equation

$$\left(r(t) \mid x'(t)\mid^{\gamma-1}x'(t)\right)' + q(t) \mid x(t)\mid^{\lambda-1}x(t) = 0, \quad \alpha \leqslant t \leqslant \beta, \tag{1.14}$$