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The inverse along a lower triangular matrix ☆

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ABSTRACT

In this paper, we investigate the recently defined notion of inverse along an element in the context of matrices over a ring. Precisely, we study the inverse of a matrix along a lower triangular matrix, under some conditions.

1. Introduction

In this paper, R is a ring with identity. We say a is (von Neumann) regular in R if $a \in aRa$. A particular solution to $axa = a$ is denoted by a^-, and the set of all such solutions is denoted by $a\{1\}$. Given $a^-, a^= \in a\{1\}$ then $x = a^= aa^-$ satisfies $axa = a,\ xax = a$ simultaneously. Such a solution is called a reflexive inverse, and is denoted by a^+. The set of all reflexive inverses of a is denoted by $a\{1,2\}$. Finally, a is group invertible if there is $a^\# \in a\{1,2\}$ that commutes with a, and a is Drazin invertible if a^k is group invertible, for some non-negative integer k. This is equivalent to the existence of $a^D \in R$ such that $a^{k+1}a^D = a^k,\ a^D aa^D = a^D,\ aa^D = a^D a$.

We say R is a Dedekind-finite ring if $ab = 1$ is sufficient for $ba = 1$. This is equivalent to saying invertible lower triangular matrices are exactly the matrices whose diagonal elements are ring units, and in this case the matrix inverse is again lower triangular.

We will make use of the Green's relation $\mathcal{H}$ in R, see [3], defined by

$$a\mathcal{H}b \quad \text{if } aR = bR \quad \text{and} \quad Ra = Rb.$$

$b \leqslant_{\mathcal{H}} d$ denotes $b \in dR \cap Rd$.

In this paper, we will study invertibility along a fixed element, as defined recently in [7] in the context of semigroups.

Definition 1.1. Given $a,\ d$ in R, we say a is invertible along d if there exists b such that $bad = d = dab$ and $b \leqslant_{\mathcal{H}} d$. If such an element exists the it is unique and is denoted by $a^{\parallel d}$.

The inverse along an element reduces to von Neumann, group and Drazin inverses (see [7]) by $a^{\parallel 1} = a^{-1},\ a^{\parallel a} = a^\#$, $a^{\parallel a^k} = a^D$.

In this paper, the existence of $a^{\parallel d}$ by means of a unit in the ring R as studied in [8] will allow us to study invertibility of some matrices along lower triangular matrices. We will give an alternative proof for the sake of completeness. In order to do so, we state a well known preliminary result.

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Lemma 1.2 (*Jacobson*). *$1-xy$ is a unit if and only if $1-yx$ is a unit, in which case $(1-xy)^{-1} = 1 + x(1-yx)^{-1}y$.*

We refer the reader to [1,2] for a similar result with Drazin inverses.

Theorem 1.3. *Let $a, d \in R$ such that d is a regular element of a ring R, and let $d^- \in d\{1\}$. Then the following are equivalent:*

1. *$a^{\parallel d}$ exists.*
2. *$u = da + 1 - dd^-$ is a unit.*
3. *$v = ad + 1 - d^-d$ is a unit.*

In this case,

$$a^{\parallel d} = u^{-1}d, \\ = dv^{-1}.$$

Proof. (2) and (3) are equivalent by writing $u = 1 - d(d^- - a)$, $v = 1 - (d^- - a)d$ and applying Lemma 1.2.

Suppose now $a^{\parallel d}$ exists, that is, there is $b \in R$ such that $bad = d = dab$ with $b = dx = yd$, for some $x, y \in R$. Since $(dadd^- + 1 - dd^-)(dxd^- + 1 - dd^-) = 1 = (ydd^- + 1 - dd^-)(dadd^- + 1 - dd^-)$ then $dadd^- + 1 - dd^-$ is a ring unit. Note that we can write $u = dd^-da + 1 - dd^- = 1 + dd^-(1 - da)$ and therefore u is a unit if and only if $dadd^- + 1 - dd^- = 1 - (1 - da)dd^-$ is a unit, using Lemma 1.2.

Conversely, suppose u, and therefore, v are units. Since $ud = dad = dv$ then $u^{-1}d = dv^{-1}$ and $d = (u^{-1}d)ad = da(dv^{-1})$. Taking $b = u^{-1}d = dv^{-1}$ then obviously $b \in Rd \cap dR$. Therefore $a^{\parallel d} = b = u^{-1}d = dv^{-1}$. □

The previous theorem shows, in particular, that given d regular then $1^{\parallel d}$ exists if and only if $d^{\#}$ exists, using [13] and Lemma 1.2.

2. The inverse of a lower triangular matrix along another lower triangular matrix

Let D be a regular lower triangular matrix and suppose $B = A^{\parallel D}$ exists, with A lower triangular. Write $D = \begin{bmatrix} d_1 & 0 \\ d_2 & d_3 \end{bmatrix}$ and $A = \begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$. According to [9], the regularity of D is equivalent to the regularity of $w = (1 - d_3d_3^+)d_2(1 - d_1^+d_1)$ for one and hence all choices of reflexive inverses d_1^+ and d_3^+ of d_1 and d_3, respectively. Using [9], there is D^- such that

$$DD^- = \begin{bmatrix} d_1d_1^+ & 0 \\ (1 - ww^-)(1 - d_3d_3^+)d_2d_1^+ & d_3d_3^+ + ww^-(1 - d_3d_3^+) \end{bmatrix}.$$

Consider now the matrix

$$U = DA + I - DD^- = \begin{bmatrix} d_1a + 1 - d_1d_1^+ & 0 \\ d_2a + d_3b - (1 - ww^-)(1 - d_3d_3^+)d_2d_1^+ & d_3d + 1 - d_3d_3^+ - ww^-(1 - d_3d_3^+) \end{bmatrix}.$$

The existence of $A^{\parallel D}$ is equivalent to the invertibility of U.

Furthermore, using Definition 1.1 together with Theorem 1.3, if $A^{\parallel D}$ exists then

$$D(V^{-1}A)D = A^{\parallel D}AD = D = DAA^{\parallel D} = D(AU^{-1})D$$

and therefore $AU^{-1}, \ V^{-1}A \in D\{1\}$.

Lemma 2.1. *Given $M = \begin{bmatrix} m_1 & 0 \\ m_2 & m_3 \end{bmatrix}$ with regular diagonal elements, then M has a lower triangular von Neumann inverse if and only if $(1 - m_3m_3^+)m_2(1 - m_1^+m_1) = 0$.*

Proof. Writing $M = \begin{bmatrix} m_1 & 0 \\ 0 & m_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ m_2 & 0 \end{bmatrix} = A + B$, let $Y = (I - AA^+)BU^{-1}(I - A^+A)$, with $U = I + A^+B = \begin{bmatrix} 1 & 0 \\ m_3^+m_2 & 1 \end{bmatrix}$. By Hartwig et al. [5, Corollary 2.7], M has a lower triangular von Neumann if and only if $Y = 0$, that is, $(1 - m_3m_3^+)m_2(1 - m_1^+m_1) = 0$. □

Theorem 2.2. *Suppose R is Dedekind-finite and let A and D be as above. Then $A^{\parallel D}$ exists if and only of $a^{\parallel d_1}$ and $d^{\parallel d_3}$ exist and $(1 - d_3d_3^+)d_2(1 - d_1^+d_1) = 0$.*

In this case,