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Double precision rational approximation algorithm for the inverse standard normal first order loss function

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ABSTRACT

We present a double precision algorithm based upon rational approximations for the inverse standard normal first order loss function. This function is used frequently in inventory management. No direct approximation or closed formulation exists for the inverse standard normal first order loss function. Calculations are currently based on root-finding methods and intermediate computations of the cumulative normal distribution or tabulations. Results then depend on the accuracy and valid range of that underlying function. We deal with these issues and present a direct, double precision accurate algorithm valid in the full range of double precision floating point numbers.

1. Introduction and motivation

Let φ be the standard normal probability density function, Φ the standard normal cumulative distribution and Φ_0 its complementary function, as respectively defined by (1), (2) and (3a). Then Φ_1 is the standard normal first order loss function, see (4a).

The inverse of the standard normal cumulative distribution, Φ_{0inv}, is defined by (3b). Wichura [1] provides an approximation algorithm for Φ_{0inv}. The focus of this article is on the inverse standard normal first order loss function Φ_{1inv}, see (4b).

$$\varphi(z) = \frac{exp(-z^2/2)}{\sqrt{2\pi}}, \tag{1}$$

$$\Phi(z) = \int_{-\infty}^{z} \varphi(x)dx, \tag{2}$$

$$\Phi_0(z_{p0}) = \int_{z_{p0}}^{\infty} \varphi(x)dx = p_0, \tag{3a}$$

$$\Phi_{0inv}(p_0) = z_{p0}, \tag{3b}$$

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$$\Phi_1(z_{p1}) = \int_{z_{p1}}^{\infty} (x - z_{p1})\varphi(x)dx = \int_{z_{p1}}^{\infty} \Phi^0(x)dx = p_1, \tag{4a}$$

$$\Phi_{1inv}(p_1) = z_{p1}. \tag{4b}$$

1.1. A brief literature review

The cumulative normal distribution and its approximation have been studied in several papers. Waissi and Rossin [2] presented a simple sigmoid function for the approximation of the cumulative standard normal probabilities for $-8 \leqslant z \leqslant 8$. Bryc [3] presented two simple formulas for the approximation of the standard normal right tail probabilities. Kiani et al. [4] worked out a formula and series for approximating the normal distribution. This formula has a maximum absolute error of 6.5e−9 and the series have a very high accuracy over the whole range. Linhart [5] compared three C functions to compute the logarithm of the cumulative standard normal distribution, based upon existing algorithms.

In Fig. 1 the functions Φ_{0inv} and Φ_{1inv} are plotted over the p range $[0.001, 0.999]$.

A closed formulation for computing Φ_1 does not exist. Shore [6] provides a simple but very rough approximation for Φ_1 and calls it the loss integral of the normal distribution. Calculating the values of the standard normal first order loss function can be done using the cumulative normal distribution, see (5), see also (1.12) of Withers and McGavin [7]. Although Gallego et al. [8] state that all standard software packages, including MS Excel, have a built in function for Φ, this holds risks because the Φ accuracy can be bad and the valid range limited, as discussed in West [9]. So currently two intermediate step are needed to calculate Φ_{1inv}. First Φ_1 needs to be calculated, and the accuracy of the result depends on the accuracy and valid range of the underlying cumulative normal function. Secondly also a root-finding method is necessary on top of this, to calculate Φ_{1inv}. With the approximations developed in this paper we want to deal with these issues.

$$\Phi_1(z) = -z\Phi_0(z) + z\varphi(z). \tag{5}$$

1.2. Inventory performance measures

In inventory management a set of performance measures are used to express the quality of the policy. A well known performance measure is the stockout frequency $\overline{A}$ (percentage of time there is no stock). In case of a normal demand this measure can be expressed using the normal cumulative distribution and its loss function. The stockout frequency expression may vary depending on the replenishment policy. We will refer here to the (r, Q) policy in which an order of size Q is placed as soon as the inventory position falls to or below the reorder point r. In Section 4.9 from Hadley and Within [10] the normal demand approximation is already discussed and they make use of (5) to express the considered performance measure and an additional root-finding method is assumed to compute the inverse function. Zipkin [11] expresses in Section 6.4.3 the stockout frequency as (6) and this can be simplified in practice for most cases to (7). So computing the optimal reorder point r value requires the function Φ_{1inv}.

$$\overline{A} = \frac{\sigma}{Q}\left[\Phi_1(z_{(r)}) - \Phi_1(z_{(r+Q)})\right], \tag{6}$$

$$\overline{A} = \frac{\sigma}{Q}\left[\Phi_1(z_{(r)})\right]. \tag{7}$$

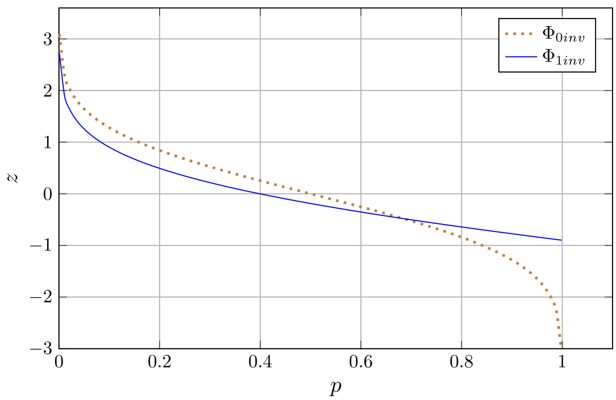

Fig. 1. Φ_{0inv} and Φ_{1inv} over range [0.001, 0.999].