Applied Mathematics and Computation 218 (2012) 6711–6715

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Hyperchaotic states in the parameter-space

Marcos J. Correia, Paulo C. Rech *

Departamento de Física, Universidade do Estado de Santa Catarina, 89219-710 Joinville, Brazil

ARTICLE INFO

Keywords:
Parameter-space
Lyapunov exponents
Chaos
Hyperchaos
Nonlinear differential equations

ABSTRACT

In this paper we propose a numerical method to characterize hyperchaotic points in the parameter-space of continuous-time dynamical systems. The method considers the second largest Lyapunov exponent value as a measure of hyperchaotic motion, to construct two-dimensional parameter-space color plots. Different levels of hyperchaos in these plots are represented by a continuously changing yellow–red scale. As an example, a particular system modeled by a set of four nonlinear autonomous first-order ordinary differential equations is considered. Practical applications of these plots include, by instance, walking in the parameter-space of hyperchaotic systems along desirable paths.

1. Introduction

Historically, hyperchaos was first presented by Rössler [1] to characterize a chaotic system with more than one positive Lyapunov exponent. It means that the dynamics of the system is expanded in two or more directions simultaneously, resulting in a more complex chaotic attractor when we compare with the chaotic system with only one positive Lyapunov exponent. This expansion of the dynamics, happening at the same time in two or more directions, makes hyperchaotic systems have better performance in many chaos based fields, including technological applications, when compared to chaotic systems. For example, hyperchaotic systems, due to higher unpredictability and the much more complicated structure of the attractors, can be used to improve the security in chaotic communication systems, where a chaotic signal is used to mask the message to be transmitted, once messages masked by chaotic systems are not always secure [2].

Hyperchaotic systems are common in many fields such as nonlinear circuits [3,4], secure communications [5,6], lasers [7,8], colpitts oscillators [9], control [10–12], synchronization [13–17], quantum cellular neural network [18], and so on. A standard way to investigate the dynamics of a hyperchaotic system is by modeling them with differential equations. If we prefer to use autonomous first order differential equations to model a hyperchaotic system, we need to consider at least four these equations. As is well known [19], in a hyperchaotic four-dimensional dissipative system there is only one possibility to the Lyapunov spectrum: two exponents are positive, one is null, and one is negative.

A standard way widely used to investigate hyperchaotic states is by considering plots of the Lyapunov exponents as a function of only one of the parameters. Regions in these plots where the largest and the second largest Lyapunov exponents are simultaneously greater than zero, characterize a hyperchaotic behavior of a particular system. Therefore, in the case of a n-parameter system, $(n-1)$ parameters are always kept fixed, and only one is varied. In this paper we propose a method that also considers the magnitude of the second largest Lyapunov exponent to numerically characterize points with hyperchaotic behavior, in which two parameters are simultaneously varied. Therefore, the method uses a two-dimensional parameter-space of a dynamical system modeled by a set of nonlinear autonomous first-order ordinary differential equations. Each point on this parameter-space is painted with a color that indicates the level of hyperchaos of the point. Here we report

* Corresponding author.
E-mail address: dfi2pcr@joinville.udesc.br (P.C. Rech).

0096-3003/$ - see front matter
doi:10.1016/j.amc.2011.12.035

specific results obtained for a prototype, which is a particular four-dimensional system constructed by us from a three-dimensional set of nonlinear autonomous first-order ordinary differential equations proposed by Wang [24], by introducing a state feedback control to the first equation. This new controlled system is given by

$$\begin{aligned} \dot{x} &= a(x-y) - yz + w, \\ \dot{y} &= -by + xz, \\ \dot{z} &= -cz + dx + xy, \\ \dot{w} &= -e(x+y), \end{aligned} \tag{1}$$

where x, y, z, w represent dynamical variables, and $a, b, c, d, e > 0$ are parameters. Here the parameters $b = 9$, $c = 5$, and $d = 0.06$ are kept fixed, while a and e are simultaneously varied.

2. Some elementary properties of the new system

The divergence of the vector field (1) is given by

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{w}}{\partial w} = a - b - c = a - 14, \tag{2}$$

from where we conclude that the system (1) is dissipative for $a < 14$ and, therefore, that the phase-space contracts volumes as the time increases. As a consequence, all the bounded system trajectories finally settle onto an attractor in a four-dimensional phase-space, by choosing adequately the parameter range $0 < a < 14$.

The equilibrium points of the system (1) are calculated by doing $\dot{x} = \dot{y} = \dot{z} = \dot{w} = 0$, that is, by solving the set of coupled equations

$$\begin{aligned} &a(x-y) - yz + w = 0, \\ &-by + xz = 0, \\ &-cz + dx + xy = 0, \\ &-e(x+y) = 0, \end{aligned} \tag{3}$$

for x, y, z, and w. Obviously the origin $P_0 = (0,0,0,0)$ is an equilibrium point. Other two equilibria are $P_1 = ((d+\beta)/2, -(d+\beta)/2, -b, \beta(b/2 - a) + bd/2 - ad)$ and $P_2 = ((d-\beta)/2, -(d-\beta)/2, -b, -\beta(b/2 - a) + bd/2 - ad)$, where $\beta = \sqrt{d^2 + 4bc}$. The Jacobian matrix for system (1) at P_0, denoted by J, is given by

$$J = \begin{pmatrix} a & -a & 0 & 1 \\ 0 & -b & 0 & 0 \\ d & 0 & -c & 0 \\ -e & -e & 0 & 0 \end{pmatrix},$$

and the characteristic equation, calculated using $\det(J - mI) = 0$, where m represents the eigenvalues and I is the 4×4 identity matrix, is

$$(m+c)(m+b)(m^2 - am + e) = 0, \tag{4}$$

from where we obtain the eigenvalues

$$m_1 = -b, \quad m_2 = -c, \quad \text{and} \quad m_\pm = \frac{a \pm \sqrt{a^2 - 4e}}{2}.$$

As is well known [19], the origin P_0 is a stable equilibrium point if the real part of the corresponding eigenvalues is negative. Take into account that $a > 0$ and $e > 0$, we conclude that if $a^2 > 4e$, m_+ is always real and positive. We also conclude that if $a^2 < 4e$, $m_\pm$ are complex conjugate with real part greater than zero. Therefore, with respect to the origin P_0, it is an unstable saddle-node equilibrium point, a necessary condition to the occurrence of chaos (or hyperchaos) in system (1).

For parameters $b = 9$, $c = 5$, and $d = 0.06$ the eigenvalues of the Jacobian matrix associated to the points P_1 and P_2 are not analytically obtained, as a function of the other parameters, a and e. They are a solution of a four-degree polynomial in m that does not factorize as Eq. (4) above. We point out that the analytical expressions of the eigenvalues are not necessary to discuss the stability of an equilibrium point. The Routh–Hurwitz criterion [19–23] can be used in order to reach this purpose. We do not present here the result of the application of the Routh–Hurwitz criterion, because our principal interest is in numerical results for the parameter-space, which appear in the next section.

3. Numerical characterization of the hyperchaotic motion

Fig. 1 shows two parameter-space plots displaying, each one of them, different dynamical behaviors for system (1). Both plots were obtained by computing Lyapunov exponents on a 500×500 mesh of parameters (e,a). In Fig. 1(a) is considered