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Optimal consumption/investment problem with light stocks: A mixed continuous-discrete time approach

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ABSTRACT

This paper addresses the optimal consumption/investment problem in a mixed discrete/continuous time model in presence of rarely traded stocks. Stochastic control theory with state variable driven by a jump-diffusion, via dynamic programming, is used. The theoretical study is validated through numerical experiments, and the proposed model is compared with the classical Merton's portfolio. Some financial insights are provided.

1. Introduction

In finance one of the most debated issues is the optimal asset allocation, which is particularly relevant both by a theoretical and a practical perspective.

The pioneer of portfolio theory is [10], which proposes a single period model with normally distributed risky returns and absence of transaction costs. The architecture of the Markowitz's model has been reviewed in a more realistic fashion by the Markowitz's followers.

Samuelson [19] extends the original uniperiodal framework to a multiperiod setting, while [12,13] deal with a continuous time portfolio model.

Continuous time portfolio models have been improved by the introduction of random jumps in the dynamics of the risky assets (see [1,8]) where, in some cases, the presence of jumps depends on transaction costs (see [9]).

In this paper, the optimal consumption/investment problem is addressed in a mixed continuous-discrete-time model, in order to consider rarely traded assets.

The frequency of trade is a measure of the stock liquidity, so that an infrequently traded stock is associated to low liquidity. Hereafter, we refer to *thin* or *light* stocks as synonymous of *rarely traded* assets.

The problem discussed here is relevant both by a theoretical and a financial point of view. By a financial perspective, the presence of thin securities is a widespread phenomenon which becomes even more relevant when risky assets of emerging economies are considered. In this regard, it is worth noting that significant relations between low trading volumes and low market quality (i.e. wide bid/ask spreads, high volatility, low informative efficiency, high adverse selection costs), documented by several empirical studies (see [7]), become even more important in times of financial crisis. By a mathematical perspective, continuous-time hypothesis may realistically describe the dynamics of a high-liquidity risky asset, but it becomes unreasonable when thinly traded assets are taken into account. Therefore the introduction of discrete-time random

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dynamics for the returns of thin stocks is required. In this paper random trading times are considered, and this leads to a very complex model.

Some other papers discuss the problem highlighted above. Matsumoto [11] introduces trades at random times following a Poisson law, but consumption is not taken into account. Rogers [17] introduces discrete random times, but consumption is assumed to be constant between trading dates. In the stochastic optimal control problem of [18], the authors introduce consumption with a rate changing between trading dates, but the optimal consumption policy is not derived in a closed form. Cretarola et al. [4] limits its analysis to the case of choice between illiquid assets and consumption.

The contribution of this paper relies on the presence of a time dependent stochastic consumption, in a financial setting with riskless bonds, frequently traded risky assets and thin stocks. In doing so, the papers quoted above are extended in many directions. According with [4], we adopt the model developed by Pham and Tankov [15] for describing the dynamics of infrequently traded assets. In particular, jumps are modeled by using a Levy process, which is particularly appropriate for this purpose (see [2,3,5,6,20]).

Stochastic control theory, in a dynamic programming framework with jump diffusions, is the followed approach. For a survey on stochastic control theory with state variables driven by jump diffusions, see [14].

The value function of the control problem is given by the maximized discounted expected utility of the investor. The problem is first theoretically solved in a very general setting; then, focusing on a particular power-type utility function, the optimal strategies in explicit closed form are derived. The solving strategy is in line with the approach adopted by Shin et al. [21], which solves a general consumption/investment problem with downside constraints and uses a CRRA utility function to deal with the numerical validation of the theoretical model.

The optimal strategies are then compared to the ones of the classical [13]'s model. Such a comparison is particularly interesting, since Merton's seminal work deals with an optimal consumption/investment problem where there are not opportunities to invest in low-liquidity stocks. The numerical validation of the theoretical model provides insights on the optimal strategies and paths in relation to the frequency of the trading dates in the thin stock.

The paper is organized as follows. Next section presents the development of the model in a general framework. Section 3 provides the analysis of the optimal strategies with a power-type utility function. Section 4 is devoted to some numerical experiments and presents the comparison between the proposed theoretical model and Merton's one. Last section concludes.

2. Model development in a general setting

In this section the economic framework of the model is presented. All the random quantities defined throughout the paper are assumed to be contained in a probability space with filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geqslant 0}, P)$, where the filtration $\mathcal{F}_t$ is assumed to reflect the whole set of information provided by the market up to time t. The investor shares her/his wealth among three assets, i.e. a risk free bond, a liquid risky asset and a light stock:

- the price of the riskless bond B_t evolves according to the following ordinary differential equation:

$$dB_t = r(t)B_t dt, \quad t \geqslant 0, \tag{1}$$

where $r(t)$ is the deterministic continuously compounded risk free interest rate at time t;
- the price of the risky liquid asset S_t evolves as follows:

$$dS_t = \mu_1 S_t dt + \sigma_1 S_t dW_t^1, \quad t \geqslant 0, \tag{2}$$

where the expected rate of return μ_1 is greater than $r(t)$, for each $t \geqslant 0$, and σ_1 is the instantaneous standard deviation of the rate of return; W^1 is a standard 1-dimensional Brownian Motion;
- the price of the thin stock H_t is assumed to follow a geometric Brownian Motion:

$$dH_t = \mu_2 H_t dt + \sigma_2 H_t dW_t^2, \quad t \geqslant 0, \tag{3}$$

where the expected rate of return μ_2 is greater than $r(t)$, for each $t \geqslant 0$, and σ_2 is the instantaneous standard deviation of the rate of return of the light stock; W^2 is a standard 1-dimensional Brownian Motion.

By definition of thin stock, it is realistically assumed that $\mu_2 > \mu_1$ and $\sigma_2 > \sigma_1$.

The financial characteristics of the thin stock imply that the dynamics of its returns should be modeled by a jump-type process. The model proposed by Pham and Tankov [15] is adopted, and it is assumed that investors can trade the thin stock only at random times $\{\tau_k\}_{k \geqslant 0}$, with $\tau_0 = 0 < \tau_1 < \cdots < \tau_k < \cdots$. We denote by Z_k the stochastic return of the light stock in the random time interval $\tau_k - \tau_{k-1}$, for each $k \in \mathbb{N}$: