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A novel method for solving a class of singularly perturbed boundary value problems based on reproducing kernel method

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ABSTRACT

In this paper, a novel method is presented for solving a class of singularly perturbed boundary value problems. Firstly the original problem is reformulated as a new boundary value problem whose solution does not change rapidly via a proper transformation; then the reproducing kernel method is employed to solve the boundary value new problem. Numerical results show that the present method can provide very accurate analytical approximate solutions.

1. Introduction

Singularly perturbed problems arise frequently in applications including geophysical fluid dynamics, oceanic and atmospheric circulation, chemical reactions, optimal control, etc. These problems are characterized by the presence of a small parameter that multiplies the highest order derivative, and they are stiff and there exist boundary layers where the solutions change rapidly.

Such problems have attracted much attention. The numerical treatment of singularly perturbed problems present some major computational difficulties, and in recent years a large number of special-purpose methods have been proposed to provide accurate numerical solutions [1–7].

Reproducing kernel theory has important applications in numerical analysis, differential equations, probability and statistics and so on [8–19]. Recently, using the reproducing kernel method (RKM), the authors have discussed various differential equations [12–19]. However, it is very difficult to expand the application of the RKM to singularly perturbed boundary value problems.

In this paper, based on the RKM, a new method is presented for the following singularly perturbed problem

$$\begin{cases} \varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = g(x), \quad 0 < x < 1, \\ u(0) = 0, \quad u(1) = 0, \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$, $a(x)$, $b(x)$ and $g(x)$ are assumed to be sufficiently smooth, and $a(x) \geqslant \alpha > 0$, α is a constant. Under the above assumption, (1.1) has a solution with a boundary layer at $x = 0$. Here we only consider $u(0) = u(1) = 0$ since the boundary conditions $u(0) = \alpha$, $u(1) = \beta$ can be reduced to $u(0) = u(1) = 0$.

The rest of the paper is organized as follows. In the next Section, a transformation is introduced. The RKM for the transformed problem is presented in Section 3. The numerical results are given in Section 4. Section 5 ends this paper with a brief conclusion.

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2. Transformation of the problem (1.1)

By the singular and invertible transformation

$$x = e^{1-\frac{1}{s}} \quad \text{or} \quad s = \frac{1}{1-\ln(x)}$$

and introducing a new unknown function $v(s) = u(x(s))$, one obtains

$$u' = v' \Big/ \frac{dx}{ds} = s^2 e^{1/s-1} v'$$

and

$$u'' = s^4 e^{2/s-2} v'' + s^2(2s-1)e^{2/s-2} v'.$$

Thus, (1.1) becomes

$$\begin{cases} a_0(s)v''(s) + a_1(s)v'(s) + a_2(s)v(s) = f(s), & 0 < s < 1, \\ v(0) = 0, \quad v(1) = 0, \end{cases} \tag{2.1}$$

where $a_0(s) = \varepsilon s^4 e^{2/s-2}$, $a_1(s) = \varepsilon s^2(2s-1)e^{2/s-2} + s^2 e^{1/s-1} a(e^{1-1/s})$, $a_2(s) = b(e^{1-1/s})$, $f(s) = g(e^{1-1/s})$.

The above transformation eliminates the boundary layer of (1.1), while generates singularity in the main part of (2.1) by the multiplier s^4. Hence, the transformed equation has the variable degeneracy of the main part. It makes this way not applicable within the framework of step-type and classical projective methods, but it is admissible for the RKM. In the next section, the RKM for transformed problem (2.1) shall be introduced.

3. Reproducing kernel method for the transformed Eq. (2.1)

In order to solve (2.1) using the RKM presented in [8,12], first, we construct a reproducing kernel space $W_2^3[0,1]$ in which every function satisfies the homogenous boundary conditions of (2.1).

The reproducing kernel Hilbert space $W_2^3[0,1]$ is defined as $W_2^3[0,1] = \{u(x)|u''(x)$ is an absolutely continuous real value functions, $u'''(x) \in L^2[0,1]$, $u(0) = 0$, $u(1) = 0\}$. The inner product and norm in $W_2^3[0,1]$ are given, respectively, by

$$(u(y), v(y))_{W_2^3} = u(0)v(0) + u'(0)v'(0) + u(1)v(1) + \int_0^1 u''' v''' dy$$

and

$$\|u\|_{W_2^3} = \sqrt{(u,u)_{W_2^3}}, \quad u, v \in W_2^3[0,1].$$

By [8,12], it is easy to obtain the reproducing kernel

$$k(x,y) = \begin{cases} k_1(x,y), & y \leqslant x, \\ k_1(y,x), & y > x, \end{cases} \tag{3.1}$$

where $k_1(x,y) = -\frac{1}{120}(x-1)y(yx^4 - 4yx^3 + 6yx^2 + (y^4 - 5y^3 - 120y + 120)x + y^4)$.

For (2.1), letting $Lv(s) = a_0(s)v''(s) + a_1(s)v'(s) + a_2(s)v(s)$, it is clear that $L : W_2^3[0,1] \to W_2^1[0,1]$ is a bounded linear operator. Put $\varphi_i(s) = \bar{k}(s_i, s)$ and $\psi_i(s) = L^*\varphi_i(s)$ where $\bar{k}(s_i, s)$ is the reproducing kernel of reproducing kernel space $W_2^1[0,1]$, L^* is the adjoint operator of L. The orthonormal system $\{\bar{\psi}_i(s)\}_{i=1}^{\infty}$ of $W_2^3[0,1]$ can be derived from Gram–Schmidt orthogonalization process of $\{\psi_i(s)\}_{i=1}^{\infty}$,

$$\bar{\psi}_i(s) = \sum_{k=1}^{i} \beta_{ik}\psi_k(s), \quad (\beta_{ii} > 0, i = 1, 2, \ldots). \tag{3.2}$$

By the RKM presented in [8,12], we have the following theorem.

Theorem 3.1. *For (2.1), if $\{s_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then $\{\psi_i(s)\}_{i=1}^{\infty}$ is the complete system of $W_2^3[0,1]$ and $\psi_i(t) = L_t k(s,t)|_{t=s_i}$.*

Proof. Note that

$$\psi_i(s) = (L^*\varphi_i)(s) = ((L^*\varphi_i)(t), k(s,t)) = (\varphi_i(t), L_t k(s,t)) = L_t k(s,t)|_{t=s_i}.$$

Clearly, $\psi_i(s) \in W_2^3[0,1]$.

For each fixed $u(s) \in W_2^3[0,1]$, let $(u(s), \psi_i(s)) = 0$, $(i = 1,2,\ldots)$, which means that,

$$(u(s), (L^*\varphi_i)(s)) = (Lu(\cdot), \varphi_i(\cdot)) = (Lu)(s_i) = 0. \tag{3.3}$$