Applied Mathematics and Computation 218 (2011) 4276–4294

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A combined nodal continuous–discontinuous finite element formulation for the Maxwell problem

Santiago Badia [a,*], Ramon Codina [b]

[a] Centre Internacional de Metodes Numerics en Enginyeria (CIMNE), Parc Mediterrani de la Tecnologia, Universitat Politècnica de Catalunya, Esteve Terrades 5, 08860 Castelldefels, Spain
[b] Universitat Politècnica de Catalunya, Jordi Girona 1–3, Edifici C1, 08034 Barcelona, Spain

ARTICLE INFO

Keywords:
Maxwell's problem
Electromagnetism
Finite elements
Stabilization techniques
Discontinuous Galerkin methods
Singular solutions

ABSTRACT

Continuous Galerkin formulations are appealing due to their low computational cost, whereas discontinuous Galerkin formulation facilitate adaptative mesh refinement and are more accurate in regions with jumps of physical parameters. Since many electromagnetic problems involve materials with different physical properties, this last point is very important. For this reason, in this article we have developed a combined cG–dG formulation for Maxwell's problem that allows arbitrary finite element spaces with functions continuous in patches of finite elements and discontinuous on the interfaces of these patches. In particular, the second formulation we propose comes from a novel continuous Galerkin formulation that reduces the amount of stabilization introduced in the numerical system. In all cases, we have performed stability and convergence analyses of the methods. The outcome of this work is a new approach that keeps the low CPU cost of recent nodal continuous formulations with the ability to deal with coefficient jumps and adaptivity of discontinuous ones. All these methods have been tested using a problem with singular solution and another one with different materials, in order to prove that in fact the resulting formulations can properly deal with these problems.

1. Introduction

The simulation of electromagnetic phenomena increasingly demands more complex geometries and larger scale problems. As an example, Maxwell's equations are in the core of plasma physics and magnetohydrodynamics simulations, coupled with the Vlasov and Navier–Stokes equations respectively. Finite element (FE) methods are a popular approach for the numerical simulation of Maxwell's systems, since realistic geometries can be handled via unstructured grids and can be straightforwardly applied to multi-physics. Furthermore, they posses a strong mathematical foundation that allows one to analyze stability and convergence properties of the algorithms.

Maxwell's equations can be casted in a saddle-point mathematical structure, with the particular feature that the multiplier is zero. With a crude Galerkin formulation, we are enforced to use inf–sup stable FE spaces; for this particular problem, this type of elements are the so-called Nédélec's FEs for the magnetic field and continuous Lagrangian interpolations for the multiplier. Unfortunately, exact penalty formulations that eliminate the Lagrange multiplier from the equations and allow FE spaces that do not satisfy the inf–sup condition lead to spurious solutions (see [17]). In order to rehabilitate the exact penalty formulation, some techniques have been proposed, e.g. the weighted penalty formulation in [17] or the decomposition of singular and smooth parts of the solution in [2,23]. Unfortunately, these approaches require to know where the singularities

* Corresponding author.
E-mail addresses: sbadia@cimne.upc.edu (S. Badia), ramon.codina@upc.edu (R. Codina).

0096-3003/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2011.09.058

will appear (re-entrant corners) and so, are hard to use in an automatic and general way. Very recently, the authors have considered an alternative way to circumvent the inf–sup condition in [4]; see [7] for a related approach. We still consider a saddle-point structure, but now using an equivalent augmented formulation and a stabilized FE discretization of the problem. This way, the resulting method allows one to capture singular solutions for continuous nodal (Lagrangian) FE spaces that do not satisfy the discrete inf–sup condition of the original system. Further, the method is very easy to implement, essentially general and automatically applicable to any problem, without the need to know where the singularities will take place.

Nodal FE approximations are very appealing in terms of CPU cost *versus* accuracy and by the simplicity of data bases (both the solution and Lagrange multiplier can be approximated using the same FE space). Furthermore, they are very effective when considering multi-physics problems, since all the unknowns of the different problems can be solved using the same finite dimensional spaces. For instance, in the magnetohydrodynamics (MHD) problem, the Navier–Stokes equations are often solved using nodal FE stabilization techniques. Analogously, the Vlasov equations in plasma physics can be solved this way. It is clear that this kind of approach greatly simplifies multi-physics codes and monolithic solvers.

However, continuous Lagrangian formulations, onwards denoted as cG, have some drawbacks that can be solved using discontinuous nodal formulations, indicated as dG formulations, that enforce continuity weakly. Examples of dG formulations for Maxwell's problem can be found in [27,24]. dG approaches are expensive but allow local mesh adaptation and are more accurate in regions in which the solution exhibits jumps, i.e. regions with jumps of physical parameters. Since many electromagnetic problems involve materials with different physical properties, this last point is very important. For this reason, in this article we have developed a combined cG–dG formulation that allows arbitrary FE spaces with functions continuous in patches of FEs and discontinuous on the interfaces of these patches; this is the sense in which the term "combined" is used in this article. This way, we are able to restrict weak continuity to a very reduced number of nodes (e.g. material interfaces or refined regions). As a result, the cG–dG approach shares the low CPU cost of cG formulations with the ability to deal with adaptivity and different materials. Analogously, the coupling of cG and dG methods has been considered in other applications (see, e.g. [26]).

The outline of the paper is as follows. In Section 2 we state the continuous problem and consider different functional settings, as well as the cG formulation in [4] and the dG one in [24]. The combined cG–dG formulation is introduced in Section 3. We have considered two alternative formulations, the second one coming from a new cG formulation based on projections that introduces le numerical dissipation. In all cases, we have performed stability and convergence analyses. Finally, in Section 4 we present some numerical experiments for a problem with singular solution and another one with discontinuous coefficients.

2. Problem statement

2.1. Notation

Let Ω be a bounded domain in $\mathbb{R}^d$, with $d = 2,3$ the space dimension. Given a Banach space X, we denote its associated norm by $\|\cdot\|_X$; for the sake of conciseness, we will omit the subscript for $L^2(\Omega)$, the space of square integrable functions. The space of vector-valued functions with components in X is denoted by X^d. The dimension superscript will be omitted in the norm, i.e. we will simply denote its norm by $\|\cdot\|_X$ instead of $\|\cdot\|_{X^d}$. The dual space of X is denoted as X'. The inner product between two scalar or vector functions $f_1, f_2 \in L^2(\Omega)$ is denoted by (f_1,f_2), whereas $\langle f_1,f_2\rangle$ is used for a duality pairing in $X \times X'$ based on the integral.

$W^{s,m}(\Omega)$ is used for the standard Sobolev space, with real coefficients $s \geqslant 0$ and $m \geqslant 1$. Hilbert spaces $W^{s,2}(\Omega)$ are denoted by $H^s(\Omega)$. We write $H_0^1(\Omega)$ for the space of functions in $H^1(\Omega)$ with null trace on $\partial\Omega$. We will make use of the following spaces of vector fields:

$$H(\mathrm{div};\varepsilon;\Omega) := \left\{\mathbf{v} \in L^2(\Omega)^d \text{ such that } \nabla\cdot(\varepsilon\mathbf{v}) \in L^2(\Omega)\right\},$$
$$H(\mathbf{curl};\Omega) := \left\{\mathbf{v} \in L^2(\Omega)^d \text{ such that } \nabla\times\mathbf{v} \in L^2(\Omega)^d\right\}$$

and the subspaces

$$H(\mathrm{div}_g;\varepsilon;\Omega) := \{\mathbf{v} \in H(\mathrm{div};\varepsilon;\Omega) \text{such that } \nabla\cdot(\varepsilon\mathbf{v}) = g\},$$
$$H_0(\mathbf{curl};\Omega) := \{\mathbf{v} \in H(\mathbf{curl};\Omega) \text{such that } \mathbf{n}\times\mathbf{v} = \mathbf{0} \text{ on } \partial\Omega\},$$

for $g \in L^2(\Omega)$ and ε a piecewise function on Ω (see below). We use the notation $A \lesssim B$ to indicate that $A \leqslant C\,B$, where A and B are expressions depending on functions that in the discrete case may depend on the discretization as well, and C is a positive constant.

2.2. The continuous problem

We consider the Maxwell problem, which physically describes magnetostatics and electrostatics in a bounded domain Ω surrounded by a perfect conductor. Let us consider $\Omega \subset \mathbb{R}^d$ to be a simply connected in general non-convex polyhedral