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Singular solutions of perturbed logistic-type equations

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ABSTRACT

We are concerned with the qualitative analysis of positive singular solutions with blow-up boundary for a class of logistic-type equations with slow diffusion and variable potential. We establish the exact blow-up rate of solutions near the boundary in terms of Karamata regular variation theory. This enables us to deduce the uniqueness of the singular solution.

1. Introduction

Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$, $N \geqslant 1$. Assume $f:[0,\infty)\to[0,\infty)$ is a locally Lipschitz continuous function such that

$$f(0)=0 \quad \text{and} \quad f(t)>0 \quad \text{for } t>0 \tag{1}$$

and

$$f \text{ is nondecreasing.} \tag{2}$$

Consider the basic population model described by the logistic problem

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega, \\ \lim_{x\to\partial\Omega} u(x) = +\infty, & \\ u>0 & \text{in } \Omega. \end{cases} \tag{3}$$

All smooth functions satisfying problem (3) are called *large* (or *blow-up boundary*) solutions.

Under assumptions (1) and (2), Keller [13] and Osserman [17] proved that problem (3) has a solution if and only if

$$\int^{+\infty} \frac{1}{\sqrt{F(u)}}\,du < +\infty, \tag{4}$$

where $F(u) := \int_0^u f(s)ds$.

We refer to Ghergu and Rădulescu [10, Theorem 1.1] for an elementary argument that problem (3) cannot have any solution if f has a sublinear or a linear growth, hence it does not satisfy condition (4). We point out that the original approach is due to Dumont et al. [8], who removed the monotonicity assumption (2) and showed that the key role in the existence of solutions of problem (3) is played only by the *Keller–Osserman condition* (4).

Functions satisfying the Keller–Osserman condition have a super-linear growth, such as: (i) $f(u) = u^p$ $(p > 1)$; (ii) $f(u) = e^u$; (iii) $f(u) = u^p \ln(1 + u)$ $(p > 1)$; (iv) $f(u) = u \ln^p(1 + u)$ $(p > 2)$.

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We point out that the study of large solutions was initiated by Bieberbach [2] in 1916 and Rademacher [19] in 1943 for the special case $f(u) = e^u$ if $N = 2$ or $N = 3$. An important contribution to the study of singular solutions with boundary blow-up is due to Loewner and Nirenberg [15], who linked the uniqueness of the large solution to the growth rate at the boundary. Motivated by certain geometric problems, they established the uniqueness of the solution in the case $f(u) = u^{(N+2)/(N-2)}$, $N \geqslant 3$.

Cîrstea and Rădulescu studied in [5] (see Du and Guo [7] for the quasilinear case) the perturbed logistic problem

$$\begin{cases} \Delta u + au = b(x)f(u) & \text{in } \Omega, \\ \lim_{x\to\partial\Omega} u(x) = +\infty, & \\ u > 0 & \text{in } \Omega, \end{cases} \tag{5}$$

where a is a real number and $b \in C^{0,\alpha}(\overline{\Omega})$, $0 < \alpha < 1$, such that $b \geqslant 0$ and $b \not\equiv 0$ in Ω. Cîrstea and Rădulescu found the whole range of values of the parameter a such that problem (5) admits a solution and this responds to a question raised by Brezis. Their analysis includes the case where the potential $b(x)$ vanishes on $\partial\Omega$. Due to the fact that u has a singular behavior on the boundary, this setting corresponds to the "competition" $0 \cdot \infty$ on $\partial\Omega$. The study carried out in [5] strongly relies on the structure of the subset of Ω where the potential b vanishes. In particular, it is argued in [5] that problem (5) has a solution for all values of $a \in \mathbb{R}$ provided that

$$\operatorname{int}\{x \in \Omega;\ b(x) = 0\} = \emptyset.$$

We also refer to Ghergu and Rădulescu [11] for related results.

Our main purpose in this paper is to study the effect of a *sublinear* perturbation au^p ($0 < p < 1$) in problem (3). This framework corresponds to a *slow diffusion* in the population model. According to Delgado and Suárez, the assumption $0 < p < 1$ means that the diffusion, namely the rate of movement of the species from high density regions to low density ones, is slower than in the linear case corresponding to $p = 1$, which is described by problem (5).

2. Statement of the problem and main results

We start with the following example of singular logistic indefinite superlinear model. Fix $m > 1$ and consider the nonlinear problem

$$\begin{cases} \Delta w^m + aw = b(x)w^2 & \text{in } \Omega, \\ \lim_{x\to\partial\Omega} w(x) = +\infty, & \\ w > 0 & \text{in } \Omega. \end{cases} \tag{6}$$

This problem can be regarded as a model of a steady-state single species inhabiting in Ω, so $w(x)$ stands for the population density. The parameter a represents the growth rate of the species while the term $m > 1$ was introduced by Gurtin and MacCamy [12] to describe the dynamics of biological population whose mobility depends upon their density. We refer to Li et al. [14] for a study of problem (6) in the case of multiply connected domains and subject to mixed boundary conditions.

The change of variable $u = w^m$ transforms problem (6) into

$$\begin{cases} \Delta u + au^p = b(x)u^q & \text{in } \Omega, \\ \lim_{x\to\partial\Omega} u(x) = +\infty, & \\ u > 0 & \text{in } \Omega, \end{cases} \tag{7}$$

where $p = 1/m \in (0,1)$ and $q = 2/m$. As stated in the previous section, it is expected that this problem has a solution in the super-linear setting, that is, provided that $m < 2$.

In this paper we study the more general problem

$$\begin{cases} \Delta u + ag(u) = b(x)f(u) & \text{in } \Omega, \\ \lim_{x\to\partial\Omega} u(x) = +\infty, & \\ u > 0 & \text{in } \Omega, \end{cases}$$

where g has a sublinear growth and f is a function satisfying the Keller–Osserman condition such that the mapping f/g is increasing in $(0,\infty)$. To fix the ideas, we consider the model problem

$$\begin{cases} \Delta u + au^p = b(x)f(u) & \text{in } \Omega, \\ \lim_{x\to\partial\Omega} u(x) = +\infty, & \\ u > 0 & \text{in } \Omega. \end{cases} \tag{8}$$

In order to describe our main result we recall some basic notions and properties from the Karamata theory of functions with regular variation at infinity. We refer to Bingham et al. [3] and Seneta [20] for more details.