Applied Mathematics and Computation 218 (2011) 4437–4447

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Chebyshev series method for computing weighted quadrature formulas ☆

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ARTICLE INFO

Keywords:
Interpolatory quadrature formula
Rational quadrature formula
Ill-scaled integrand
Difficult pole
Modified weight function
Chebyshev series

ABSTRACT

In this paper we study convergence and computation of interpolatory quadrature formulas with respect to a wide variety of weight functions. The main goal is to evaluate accurately a definite integral, whose mass is highly concentrated near some points. The numerical implementation of this approach is based on the calculation of Chebyshev series and some integration formulas which are exact for polynomials. In terms of accuracy, the proposed method can be compared with rational Gauss quadrature formula.

1. Introduction

A common problem is the evaluation of a definite integral over a bounded interval $[a,b]$. The solution of this problem can be approached in terms of a product integration rule, specially when the integrand has singularities. The procedure is to re-write the integrand as $F(x)W(x)$, where F is the integrand and W is a non-negative weight function.

Suppose that w is a weight function on $[a,b]$, which is closely related to W in a sense to be clarified. The integral of FW can be approximated by a weighted quadrature of the form

$$\int_a^b F(x)W(x)dx = \sum_{k=1}^{n} \lambda_{n,k}(W)F(x_{n,k}) + \mathcal{E}_n(F), \tag{1}$$

where

$$\lambda_{n,k}(W) = \int_a^b \frac{P_n(x)}{(x-x_{n,k})P_n'(x_{n,k})}W(x)dx \tag{2}$$

and $P_n(x) = \prod_{k=1}^{n}(x-x_{n,k})$ is the nth (monic) orthogonal polynomial with respect to w.

In the rest of this article, we will maintain the notation used in (2). The aim is to indicate explicitly the weight function to which we refer.

To improve the accuracy of the results, a plausible strategy consists in modifying conveniently the initial factorization, say, $FW = fGW$, where $G \geqslant 0$, and the main property of f is to have no singularities.

Roughly speaking, a real-valued function is said to be a *difficult function*, if it has integrable singularities, or it ranges over very large and very small values (poor scaling). A typical case is one in which the function is meromorphic on a region $V \supset [a,b]$. In this context, the adjective *difficult* also applies to the poles of the integrand that are very close to $[a,b]$.

☆ The work of A.C.L., E.B.E. and J.R.I.G. was supported by a research grant from the Ministry of Sciences and Innovation, Spain, Project code MTM 2008-00341.
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doi:10.1016/j.amc.2011.10.024

To tackle the integration of a difficult function, the technique of selecting G as a proper rational function has been used intensively, i.e. $f = qF/p$ and $G = p/q$, where p and q are polynomials whose roots match the *difficult points* of F (cf. [5–7,9]). Monegato [9] considers $G = 1/q$, and calculates $\lambda_{n,k}(W/q)$ in terms of $\lambda_{n,k}(W)$. That approach requires the calculation of the integrals $V_i = \int_a^b x^i W(x)/q(x)dx,\ i = 0, 1$. Monegato's method is based on the partial fraction decomposition of $1/q$, and depends on the ability to calculate V_i. The main objective of [9] is to obtain accurate results when the integrand has difficult poles, and this is achieved by applying a non-linear algorithm proposed by Gautschi. A variant of Monegato's work is given in [5] where it is considered $\{G_n\}$, defined as $G_n = p_n/q_n,\ n \in \mathbb{N}$, where $\{p_n\}$ and $\{q_n\}$ are two sequences of polynomials such that $p_n/q_n = \mathcal{O}(1)$. Moreover, the roots of p_n coincide with some zeros of the integrand, and the difficult poles of the integrand are neutralized by applying an appropriate change of variables. The examples in [5] demonstrate that the zeros of the integrand can play a role, but the numerical results are similar to those in [9] when $p_n \equiv 1$. A common feature to both papers [5,9] is that the convergence of the quadrature rules is guaranteed by the following condition (cf. [10]).

$$\int_a^b W(x)^2/w(x)dx < \infty. \tag{3}$$

The purpose of this paper is to present a flexible approach to evaluate the integral of difficult functions. As a result, they have been derived several methods whose performance is illustrated mainly by means of numerical examples. In a sense, some of these methods are simpler than that of Monegato [9]. Our strategy has been based on constructing interpolatory quadrature formulas with respect to a class of varying weights.

From here onwards, we will only consider integrals on the interval $[-1, 1]$.

The paper is organized as follows.

In Section 2 we consider a modification of (1) and (2) given in terms of a sequence $\{G_nW\}$, which means that the coefficients are given by $\lambda_{n,k}(G_mW)$. Then we obtain convergence when the trio $(\{G_n\}, W, w)$ satisfies a condition weaker than (3). Section 3 deals with the case $G = K/q$, where $K \in \mathcal{L}_2(w)$, and focuses on the calculation of $\lambda_{n,k}(KW/q)$. Here we assume that G is approximated by $G_m = p_m/q$, where p_m is a partial sum of the Chebyshev series expansion of K. The problem reduces to consider $w(x) = 1/\sqrt{1-x^2}$ and $W = h_1 w$, where h_1 is a polynomial. It is clear that W is a generic weight that includes the four classical Chebyshev weight functions. It is used a technique similar to that by Berriochoa et al. [1], to prove an exact formula for $\lambda_{n,k}(KW/q)$. Unlike the method proposed by Clenshaw and Curtis [4], here we only use the Chebyshev series expansion of a factor K.

The numerical examples of Section 4 allow to appraise the effectiveness of the proposed approach. To facilitate this, we compare our results with those obtained by Gautschi [7], Monegato [9] and Deckers et al. [2]. Section 5 is devoted to the calculation of the coefficients of the Chebyshev series expansion, and Section 6 contains some remarks as a conclusion.

Throughout the paper Π denotes the space of all algebraic polynomials, $\Pi_n = \{p \in \Pi;\ \deg(p) \leqslant n\}$, and $\|f\| = \sup\{|f(x)|;\ x \in [-1, 1]\}$.

2. Convergence of modified quadrature rules

Definition 1. Let W and w be two weight functions on $[-1, 1]$, both with infinitely many points of increase. Let F be an integrable function with respect to W and assume that $F = fG$, where f is continuous, and G is a nonnegative function such that $G \in \mathcal{L}_1(W)$. Let $\{G_n\}$ be a sequence of positive and continuous functions on $(-1, 1)$, such that $G_n \in \mathcal{L}_1(W),\ n \in \mathbb{N}$, and $G_nWdx \to GWdx$, as $n \to \infty$, in the weak * topology of measures. We say that

$$\int_{-1}^{1} F(x)W(x)dx = \sum_{k=1}^{n} \lambda_{n,k}(G_mW)f(x_{n,k}) + \mathcal{E}_{n,m}(f), \tag{4}$$

is a modified interpolatory quadrature formula (*MIQF*) of order (n, m), with respect to the pair $(\{G_n\}, w)$, if

$$\lambda_{n,k}(G_mW) = \int_a^b \frac{P_n(x)}{(x - x_{n,k})P_n'(x_{n,k})} G_m(x)W(x)dx, \tag{5}$$

where $P_n(x) = \prod_{k=1}^{n}(x - x_{n,k})$ is the nth (monic) orthogonal polynomial with respect to w.

As usual, the term $\mathcal{E}_{n,m}(f)$ stands for the quadrature error.

Proposition 1. *Suppose that (4) is a MIQF with respect to $(\{G_n\}, w)$, and let n be fixed. Then*

$$\int_{-1}^{1} f(x)G_m(x)W(x)dx = \sum_{k=1}^{n} \lambda_{n,k}(G_mW)f(x_{n,k}), \tag{6}$$

holds for all $f \in \Pi_{n-1}$, and all $m \in \mathbb{N}$.

Proof. Just note that $\lambda_{n,k}(G_mW)$ is the kth coefficient of the interpolatory rule of polynomial type which corresponds to the weight G_mW. □