Applied Mathematics and Computation 218 (2011) 4514–4522

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Semilinear systems involving multiple critical Hardy–Sobolev exponents and three singular points ☆

Dongsheng Kang

School of Mathematics and Statistics, South-Central University for Nationalities, Wuhan 430074, PR China

ARTICLE INFO

Keywords:
Semilinear elliptic system
Solution
Critical exponent
Hardy inequality

ABSTRACT

In this paper, a semilinear system of elliptic equations is investigated, which involves multiple critical exponents and singular points. By variational methods and analytic techniques, the related best Hardy–Sobolev constants is studied and the existence of positive solutions is established.

1. Introduction

In this paper, we study the following elliptic system:

$$\begin{cases} Lu = \frac{\sigma\alpha}{2^*(r)}\frac{|u|^{\alpha-2}|v|^{\beta}u}{|x|^r} + \eta\frac{|u|^{2^*(s)-2}u}{|x-\xi_1|^s} + a_1u + a_2v, \\ Lv = \frac{\sigma\beta}{2^*(r)}\frac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^r} + \lambda\frac{|v|^{2^*(t)-2}v}{|x-\xi_2|^t} + a_2u + a_3v, \\ (u,v) \in H_0^1(\Omega)\times H_0^1(\Omega), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N\,(N \geqslant 3)$ is a bounded domain with the smooth boundary $\partial\Omega$ such that the points $0, \xi_1, \xi_2 \in \Omega, L := \left(-\Delta\cdot -\mu\frac{\cdot}{|x|^2}\right), \eta, \lambda, \sigma \geqslant 0, a_1, a_2, a_3 \in \mathbb{R}, 0 < r, s, t < 2, 1 < \alpha, \beta < 2^*(r) - 1, \alpha+\beta = 2^*(r), \mu < \bar{\mu}, \bar{\mu} := \left(\frac{N-2}{2}\right)^2$ is the best Hardy constant, $2^*(r)$, $2^*(s)$ and $2^*(t)$ are the critical Hardy–Sobolev exponents defined as $2^*(\tau) := \frac{2(N-\tau)}{N-2}, \tau \in [0,2), H_0^1(\Omega) =: H$ is the completion of $C_0^\infty(\Omega)$ with respect to $(\int_\Omega |\nabla\cdot|^2 dx)^{1/2}$ and $H_0^1(\Omega)\times H_0^1(\Omega) =: H\times H$ is a product space.

The energy functional of (1.1) is defined in $H\times H$ by

$$J(u,v) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 - \mu\frac{u^2+v^2}{|x|^2}\right)dx - \frac{1}{2}\int_\Omega (a_1u^2 + 2a_2uv + a_3v^2)dx$$
$$- \frac{\sigma}{2^*(r)}\int_\Omega \frac{|u|^\alpha|v|^\beta}{|x|^r}dx - \frac{\eta}{2^*(s)}\int_\Omega \frac{|u|^{2^*(s)}}{|x-\xi_1|^s}dx - \frac{\lambda}{2^*(t)}\int_\Omega \frac{|v|^{2^*(t)}}{|x-\xi_2|^t}dx.$$

Then the functional $J \in C^1(H\times H, \mathbb{R})$ and a pair of functions $(u_0, v_0) \in H\times H$ is said to be a solution of (1.1), if

$$(u_0,v_0) \neq (0,0), \quad \langle J'(u_0,v_0), (\varphi,\phi)\rangle = 0, \quad \forall(\varphi,\phi) \in H\times H,$$

where $u, v, \varphi, \phi \in H$ and $J'(u,v)$ denotes the Fréchet derivative of J at (u,v). The solution (u_0,v_0) of (1.1) is equivalent to a nonzero critical point of J and a standard elliptic argument shows that $u_0, v_0 \in C^2(\Omega\setminus\{0,\xi_1,\xi_2\}) \cap C^1(\overline{\Omega}\setminus\{0,\xi_1,\xi_2\})$.

☆ This work is supported by the National Natural Science Foundation of China (10771219).
E-mail address: dongshengkang@yahoo.com.cn

0096-3003/$ - see front matter
doi:10.1016/j.amc.2011.10.033

(1.1) is related to the Hardy and Caffarelli–Kohn–Nirenberg inequalities ([5,11]):

$$\int_{\mathbb{R}^N} \frac{|u|^2}{|x-\xi|^2}\mathrm{d}x \leqslant \frac{1}{\bar{\mu}} \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^N), \quad \xi \in \mathbb{R}^N, \tag{1.2}$$

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*(t)}}{|x-\xi|^t}\mathrm{d}x\right)^{\frac{2}{2^*(t)}} \leqslant C(t) \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^N), \quad \xi \in \mathbb{R}^N, \tag{1.3}$$

where $C(t)$ is a positive constant depending on t and (1.3) is also named as the Hardy–Sobolev inequality. By (1.2), L is positive for all $\mu < \bar{\mu}$ and has discrete spectrum σ_μ in $H_0^1(\Omega)$, and the first eigenvalue $\Lambda_1(\mu)$ is simple ([8]). By (1.2) and (1.3), the following best constant is well defined for all $0 \leqslant t < 2$ and $0 \leqslant \mu < \bar{\mu}$:

$$S_{\mu,t} := \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\left(|\nabla u|^2 - \mu\frac{u^2}{|x-\xi|^2}\right)\mathrm{d}x}{\left(\int_{\mathbb{R}^N}\frac{|u|^{2^*(t)}}{|x-\xi|^t}\mathrm{d}x\right)^{\frac{2}{2^*(t)}}}, \tag{1.4}$$

where $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to $(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x)^{1/2}$. Note that $S_{\mu,t}$ is independent of ξ and is attained by the extremal functions ([14]):

$$V^\varepsilon_{\mu,t}(x-\xi) := \varepsilon^{\frac{2-N}{2}} U_{\mu,t}\left(\frac{x-\xi}{\varepsilon}\right), \quad \forall \varepsilon > 0, \tag{1.5}$$

where

$$U_{\mu,t}(x) := \frac{\left(\frac{2(\bar{\mu}-\mu)(N-t)}{\sqrt{\bar{\mu}}}\right)^{\frac{\sqrt{\bar{\mu}}}{2-t}}}{|x|^{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}}\left(1+|x|^{\frac{(2-t)\sqrt{\bar{\mu}-\mu}}{\sqrt{\bar{\mu}}}}\right)^{\frac{N-2}{2-t}}}.$$

For all $\eta, \lambda, \sigma \geqslant 0, \eta+\lambda+\sigma > 0, 0 \leqslant r < 2, \mu < \bar{\mu}, \alpha, \beta > 1$ and $\alpha + \beta = 2^*(r)$, by the Young and Hardy–Sobolev inequalities, the following constant is well defined on $\mathcal{D} := (D^{1,2}(\mathbb{R}^N)\setminus\{0\})^2$:

$$S_{\eta,\lambda,\sigma}(\mu,r) := \inf_{(u,v)\in\mathcal{D}} \frac{\int_{\mathbb{R}^N}\left(|\nabla u|^2 + |\nabla v|^2 - \mu\frac{u^2+v^2}{|x|^2}\right)\mathrm{d}x}{\left(\int_{\mathbb{R}^N}\frac{\eta|u|^{2^*(r)}+\lambda|v|^{2^*(r)}+\sigma|u|^\alpha|v|^\beta}{|x|^r}\mathrm{d}x\right)^{\frac{2}{2^*(r)}}}. \tag{1.6}$$

In recent years, much attention has been paid to the elliptic equations involving the Hardy–Sobolev inequality (e.g. [7–10,13,18] and the references therein). However, the elliptic systems involving the Hardy–Sobolev inequality have been seldom studied, only a few related results are found in [1,3,12,17], and many important topics remain open. In this paper, we investigate (1.1), which involves multiple critical terms. Note that the strongly-coupled term $\frac{|u|^\alpha|v|^\beta}{|x|^r}$ is also critical since $\alpha + \beta = 2^*(r)$.

The following assumptions are needed in this paper:

$$(\mathcal{H}_1) N \geqslant 3, \quad \eta,\lambda,\sigma \geqslant 0, \quad \eta+\lambda+\sigma > 0, \quad 0 < r,s,t < 2, \quad 0 \leqslant \mu < \bar{\mu}, \quad \alpha,\beta > 1,$$
$$\alpha+\beta = 2^*(r), \quad a_1,a_2,a_3 > 0, \quad a_1a_3 - a_2^2 > 0, \quad 0 < \lambda_1 \leqslant \lambda_2 < \Lambda_1(\mu),$$

where λ_1 and λ_2 are the eigenvalues of the matrix $A := \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}$.

Define the quadratic form

$$Q(u,v) := (u,v)A(u,v)^T = a_1u^2 + 2a_2uv + a_3v^2.$$

Under the assumption $(\mathcal{H}_1), Q(u,v)$ is positive definite and satisfies

$$\lambda_1(u^2+v^2) \leqslant Q(u,v) \leqslant \lambda_2(u^2+v^2), \quad \forall (u,v) \in H \times H. \tag{1.7}$$

We explain several notations:

$$f_{\eta,\lambda,\sigma}(\tau) := \frac{(1+\tau^2)S_{\mu,r}}{(\eta+\sigma\tau^\beta+\lambda\tau^{2^*(r)})^{\frac{2}{2^*(r)}}}, \quad \tau > 0,$$
$$f_{\eta,\lambda,\sigma}(\tau_{\eta,\lambda,\sigma}) := \min_{\tau>0} f_{\eta,\lambda,\sigma}(\tau) > 0, \quad \sigma > 0, \tag{1.8}$$

where $\tau_{\eta,\lambda,\sigma} > 0$ is a minimal point of $f_{\eta,\lambda,\sigma}(\tau)$. If $\sigma > 0$ and $N > 6 - 2r$, then $\beta > 2^*(r) - 2$ and $\beta - 2 < 2^*(r) - 3 < 0$ and direct calculation shows that $\min_{\tau>0} f_{\eta,\lambda,\sigma}(\tau)$ must be achieved at finite $\tau_{\eta,\lambda,\sigma} > 0$. Furthermore, from the fact $f'_{\eta,\lambda,\sigma}(\tau_{\eta,\lambda,\sigma}) = 0$ it follows that $\tau_{\eta,\lambda,\sigma}$ is a root of the following equation:

$$2^*(r)\eta + \alpha\sigma\tau^\beta - \beta\sigma\tau^{\beta-2} - \lambda 2^*(r)\tau^{2^*(r)-2} = 0, \quad \tau > 0. \tag{1.9}$$