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Existence and uniqueness results for third-order nonlinear differential systems ☆

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ABSTRACT

Sufficient conditions are given for the existence of solutions to third order boundary value problems for nonlinear differential systems. Some appropriate conditions are given to guarantee that the Nagumo condition is satisfied. By constructing appropriate a lower solution–upper solution pair, a concept that is defined in this paper, the uniqueness result of the problem is also established. The emphasis here is that the differential systems has nonlinear dependence on all over order derivatives and the boundary conditions are nonlinear.

1. Introduction

Consider the third order nonlinear differential systems

$$x''' = f(t, x, x', x''), \quad 0 < t < 1, \tag{1.1}$$

together with nonlinear boundary conditions

$$x(0) = A, \quad g(x'(0), x''(0)) = 0, \quad h(x'(1), x''(1)) = 0, \tag{1.2}$$

where $x \in R^n$, $A = (a_1, a_2, \ldots, a_n)^T \in R^n$, $f \in C([0,1] \times R^{3n}, R^n)$, $g, h \in C^1(R^{2n}, R^n)$.

In the last few years, an extensive research has been made on the existence of the boundary value problem (BVP, for short) for third-order nonlinear differential equations, for instance, to see [1–3,5–7,10,11,13–18]. Many techniques arose in the studies of this kind problem. For example, upper and lower solutions method [2,5,6,9–11,18], topological transversality [1,14], Mawhin coincidence degree theory [7], monotone iterative techniques [13,15], numerical method [16,17], comparable analysis with classical equations [3].

Recently, Grossinbo and Minhós [10] studied the existence to a third order boundary value problems with linear separated boundary conditions. Du et al. [6] extended the existence result in [10] to nonlinear boundary value problems with nonlinear boundary conditions. Grossinbo et al. [11] extended the results in [6,10] to a nonlinear boundary value problem under the generated Nagumo condition. Wang [18] used the method of descent to study a third nonlinear boundary value problem, which is similar to the problem discussed in [6,11].

But all the above-mentioned papers discussed scalar differential equations, and boundary conditions are all linear except references [6,11,18]. The existence of solutions for third order nonlinear differential systems

$$x'''(t) = f(t, x(t), x'(t), x''(t)),$$

together with the following boundary conditions

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$$x(0) = A, \quad P_1x'(0) - P_1x''(0) = B, \quad Q_1x'(1) + Q_2x''(1) = C$$

was investigate in [19], by using the differential inequalities theory and the method of descent. In [8], the authors considered a second-order nonlinear differential systems

$$x'' = f(t, Sx, x, x'), \quad 0 < t < 1$$

subject to the nonlinear conditions

$$x(0) = A, \quad h(x(1), x'(1)) = B,$$

where x, f, h, A and B are n-vectors. For some existence results of second order linear separated differential systems, we refer the reader to [4,20].

However the results on third-order nonlinear differential systems are very poor and rare works are done for nonlinear boundary conditions. Motivated by the work of the above papers, in this paper, we shall employ the method of upper and lower solutions and the method of descent to study the existence of solutions of a boundary value problem for third order nonlinear differential systems (1.1) with some quite general nonlinear boundary conditions (1.2). By constructing appropriate upper and lower solutions of Eq. (1.1), the uniqueness result of solution is also investigated for BVP (1.1), (1.2).

2. Preliminary

To discuss the problem (1.1), (1.2), let $x' = u$, then BVP (1.1), (1.2) can be transformed as the following second order Volterra type integro-differential systems

$$u'' = f(t, Su, u, u'), \quad 0 < t < 1, \tag{2.1}$$

with nonlinear boundary conditions:

$$g(u(0), u'(0)) = 0, \quad h(u(1), u'(1)) = 0, \tag{2.2}$$

where $[Su](t) = (S_1u_1, \ldots, S_nu_n)$, $[S_iu_i](t) = a_i + \int_0^t u_i(s)ds$, $i = 1, 2, \ldots, n$.

We now give the Nagumo growth condition on $f(t,x,y,z)$ with respect to z.

Definition 1. $f(t,x,y,z)$ is said to satisfy Nagumo condition with respect to z, for $(t,x,y,z) \in [0,1] \times R^{3n}$, if satisfies one of the following conditions:

(i) there exist functions $\Phi_i \in C[0,+\infty)$, $i = 1,2,\ldots,n$, such that

$$|f_i(t,x,y,z)| \leqslant \Phi_i(|z_i|), \quad \int_0^{+\infty} \frac{sds}{\Phi_i(s)} = +\infty;$$

(ii) for arbitrary $(t,x,y,z) \in [0,1] \times R^{3n}$, $i = 1,2,\ldots,n$,such that

$$f_i(t,x,y,z) = O\left(|z_i|^2\right), \quad |z_i| \to +\infty,$$

where $\|\cdot\|$ is Euclid norm.

The following upper and lower solutions are used to obtain *a priori* bounds on solutions (2.1) in [8].

Definition 2. $\alpha(t) = (\alpha_1(t),\ldots,\alpha_n(t))^T$ and $\beta(t) = (\beta_1(t),\ldots,\beta_n(t))^T \in C^2([0,1],R^n)$ are called lower and upper solutions of Eq. (2.1), respectively, provided that

(i) $\alpha_i(t) \leqslant \beta_i(t)$, $0 \leqslant t \leqslant 1$, $i = 1,2,\ldots,n$;
(ii) $\alpha_i''(t) \geqslant f_i\left(t, Su_{\alpha_i}, u_{\alpha_i}, u'_{\alpha_i}\right)$, $\beta_i''(t) \leqslant f_i\left(t, Su_{\beta_i}, u_{\beta_i}, u'_{\beta_i}\right)$, $i = 1, 2, \ldots, n$,

where

$$Su_{\alpha_i} = (S_1u_1, \ldots, S_{i-1}u_{i-1}, S_i\alpha_i, S_{i+1}u_{i+1}, \ldots, S_nu_n),$$
$$u_{\alpha_i} = (u_1, \ldots, u_{i-1}, \alpha_i, u_{i+1}, \ldots, u_n),$$
$$u'_{\alpha_i} = \left(u'_1, \ldots, u'_{i-1}, \alpha'_i, u'_{i+1}, \ldots, u'_n\right),$$

for all $S_ju_j \in [S_j\alpha_j, S_j\beta_j]$, $u_j \in [\alpha_j, \beta_j]$, $j \neq i$, $i = 1, 2, \ldots, n$. Su_{β_i}, u_{β_i} and u'_{β_i} are defined analogously.