Applied Mathematics and Computation 218 (2011) 3149–3157

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Geometric integration of the paraxial equation

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ARTICLE INFO

Keywords:
Symplectic method
The paraxial equation
Energy conservation
Stability
Solitary waves
Photorefractive crystal

ABSTRACT

Evolution of solitary waves in photovoltaic–photorefractive crystal satisfy the paraxial equation. The paraxial equation is transformed into the symplectic structure of the infinite dimensional Hamiltonian system. The symplectic structure of the paraxial equation is discretizated by the symplectic method. The corresponding symplectic scheme preserves conservation of discrete energy which reflects conservation of energy of the paraxial equation. The symplectic scheme is applied to simulate the solitary wave behaviors of the paraxial equation. Evolution of the solitary waves with the different applied electric field and the different photovoltaic fields are investigated.

1. Introduction

Spatial solitons have attracted a great deal of attention because of their possible applications for optical switching and routing. Besides the electromagnetic field solitons, refractive index solitons have also been investigated in photorefractive (PR) media [1]. Several generic types of wave-field PR solitons have been predicated and observed thus far, including quasi-steady-state solitons [5], screen solitons [23], photovoltaic solitons [2], screening photovoltaic solitons [16] and spatial solitons in PR centrosymmetric materials [21] and in anisotropic nonlinear media [24], all of which result from the single photon photorefactive effect. Very recently single beam bright and dark solitons in two-photon photorefactive materials have been predicated, which result from the two-photon PR effect [11].

In one dimensional, the propagation of the solitary waves of the photovoltaic–photorefractive crystal is described by the paraxial equation [14–17,22,25,30]

$$i\frac{\partial\phi}{\partial z}+\frac{1}{2k}\frac{\partial^2\phi}{\partial x^2}-\frac{k_0}{2}\left(n_e^3 r_{eff}E_{sc}\right)\phi=0. \tag{1}$$

Here i is the imaginary unit, ϕ describes a complex envelope of the light field, $k = k_0 n_e$, $k_0 = 2\pi/\lambda_0$, λ_0 is the wavelength in the vacuum, n_e is the unperturbed refractive index and r_{eff} is the effective linear electro-optic coefficient,

$$E_{sc}=E_0\frac{I_\infty+I_d}{I+I_d}+E_p\frac{I_\infty-I}{I+I_d}-\frac{K_BT}{e}\frac{1}{I+I_d}\frac{\partial I}{\partial x} \tag{2}$$

is the space-charge field. E_0 is the applied electronic field and E_p is the photovoltaic field, K_B is constant, T is the absolute temperature, e is the unite charge, I_d is the dark radiation intensity, $I_\infty = I(x,z)|_{x\to\infty}$. Numerical simulations of the solitary waves of the paraxial equation have been investigated. In [31], three classes of (1 + 1)-D lattices solitons, including fundamental solitons, high-order lattices solitons and out-of phase dipole lattices solitons, were obtained by numerical integration

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doi:10.1016/j.amc.2011.08.048

of the theoretical model. In [29], the temporal behaviors of one single laser pulse in a photorefractive medium were simulated by the beam propagation method.

Recently, attention has been paid to the symplectic geometry [3,4,7–9,20]. A great deal of numerical experiments have shown the superiority of symplectic schemes over the nonsymplectic ones, especially, in structural, global and long-term tracking capabilities [6,12,13,18,26–28]. In this paper, the symplectic method is applied to simulate the solitary wave behaviors of the paraxial equation.

The paper is organized as follows. The symplectic scheme of the paraxial equation is obtained in Section 2. The energy conservation of the paraxial equation and of the symplectic scheme are proved in Section 3. In Section 4, the solitary wave behaviors of the paraxial equation are investigated by numerical simulations.

2. Symplectic scheme of the paraxial equation

Eq. (1) can be normalized using similar conventions as in the continuous regime. The scale transformations $t = z/\left(kx_0^2\right), s = x/x_0, u = \left(\frac{2\eta_0 I_d}{n_e}\right)^{-1/2}\phi$, where x_0 is an arbitrary spatial scale (whatever the value of the arbitrary spatial scale x_0), are adopted. A new propagation equation can derived [10,14,30], depending on the orientation of the beam electric field u,

$$iu_t + \frac{1}{2}u_{ss} - \beta(\rho+1)\frac{u}{1+|u|^2} - \alpha\frac{(\rho-|u|^2)u}{1+|u|^2} = 0, \tag{3}$$

where the applied electric field $\beta = (k_0x_0)^2\left(n_e^4 r_{eff}/2\right)E_0$ and the photovoltaic field $\alpha = (k_0x_0)^2\left(n_e^4 r_{eff}/2\right)E_p$ are constants. The boundary conditions for bright spatial solitons are: $\rho = I_\infty/I_d = 0$. Eq. (3) is equivalent to

$$iu_t + \frac{1}{2}u_{ss} - (\beta+\alpha)\frac{u}{1+|u|^2} + \alpha u = 0. \tag{4}$$

Take $u = p + iq$, Eq. (4) can be expressed as

$$p_t + \frac{1}{2}q_{ss} - \frac{(\beta+\alpha)q}{1+(p^2+q^2)} + \alpha q = 0, \tag{5}$$

$$q_t - \frac{1}{2}p_{ss} + \frac{(\beta+\alpha)p}{1+(p^2+q^2)} - \alpha p = 0. \tag{6}$$

Take $z = (p,q)^T$, Eqs. (5) and (6) can be expressed in the infinite Hamiltonian form

$$\frac{dz}{dt} = J\frac{\delta H(z)}{\delta z}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{7}$$

where $z = (p,q)$, the corresponding Hamiltonian function is

$$H(z) = \int \left[\frac{1}{2}(p_s^2+q_s^2) + \frac{\beta+\alpha}{2}In(1+(p^2+q^2)) - \frac{\alpha}{2}(p^2+q^2)\right]ds. \tag{8}$$

Now we can discrete the spatial domain of Eq. (7) and expect to obtain a finite-dimensional Hamiltonian system. As in [19], denoting the 2mth order central difference operator for $\mathcal{B} = \frac{\partial}{\partial s^2}$ by $\mathcal{B}(2m)$, we have

$$\mathcal{B}(2m) = \nabla_+\nabla_-\sum_{j=0}^{m-1}(-1)^j\beta_j\left(\frac{(\Delta s)^2\nabla_+\nabla_-}{4}\right)^j,$$

where $\beta_j = [(j!)^2 2^{2j}]/[(2j+1)!(j+1)]$ and ∇_+, ∇_- are forward and backward difference operators, and Δs is the spatial step length.

Denoting by N the number of the spatial grid points, and letting $P = [p_1, p_2, \ldots, p_N]$, $Q = [q_1, q_2, \ldots, q_N]$, $Z = (P,Q)^T$, Eq. (7) is transformed into the finite dimensional Hamiltonian system

$$\frac{dZ}{dt} = \widetilde{J}\nabla_z H(Z), \quad \widetilde{J} = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}, \tag{9}$$

with the corresponding Hamiltonian function

$$H(P,Q) = \frac{1}{4}[P^T B(2m)P + Q^T B(2m)Q] + \frac{\beta+\alpha}{2}\sum_{l=1}^{N} In\left(1+\left(p_l^2+q_l^2\right)\right) - \frac{\alpha}{2}\sum_{l=1}^{N}\left(p_l^2+q_l^2\right), \tag{10}$$

where $B(2m)$ is $N \times N$ matrix. Applying the fourth order difference scheme along the s direction, we obtain