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Iterative algorithms for the minimum-norm solution and the least-squares solution of the linear matrix equations $A_1XB_1 + C_1X^TD_1 = M_1,\ A_2XB_2 + C_2X^TD_2 = M_2$ ☆

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ABSTRACT

In this paper, two iterative algorithms are proposed to solve the linear matrix equations $A_1XB_1 + C_1X^TD_1 = M_1,\ A_2XB_2 + C_2X^TD_2 = M_2$. When the matrix equations are consistent, by the first algorithm, a solution X^* can be obtained within finite iterative steps in the absence of roundoff-error for any initial value, furthermore, the minimum-norm solution can be got by choosing a special kind of initial matrix. Additionally, the unique optimal approximation solution to a given matrix X_0 can be derived by finding the minimum-norm solution of a new matrix equations $A_1\widetilde{X}B_1 + C_1\widetilde{X}^TD_1 = M_1,\ A_2\widetilde{X}B_2 + C_2\widetilde{X}^TD_2 = M_2$. When the matrix equations are inconsistent, we present the second algorithm to find the least-squares solution with the minimum-norm. Finally, two numerical examples are tested by MATLAB, the results show that these iterative algorithms are efficient.

1. Introduction

Matrix equations are often encountered in many areas of computational mathematics, control and system theory. Research on solving linear matrix equations has been actively engaged in for many years. For example, Navarra et al. studied a representation of the general common solution of the matrix equations $A_1XB_1 = C_1,\ A_2XB_2 = C_2$ [1]; van der Woude obtained the existence of a common solution X for the matrix equations $A_iXB_j = C_{ij}$ [2]; Bhimasankaram considered the linear matrix equations $AX = B,\ CX = D$ and $EXF = G$ [3]; Mitra has provided conditions for the existence of a solution and a representation of the general common solution of the matrix equations $AX = C,\ XB = D$ and the matrix equations $A_1XB_1 = C_1,\ A_2XB_2 = C_2$ [4,5].

Traditionally, linear matrix equations can be converted into their equivalent forms by using the Kronecker product. However, in order to solve the equivalent forms, the inversion of the associated large matrix need be involved, which leads to computational difficulty because excessive computer memory is required. For this reason, iterative approaches for solving matrix equations and recursive identification for parameter estimation have always received much attention in recent years, e.g. Peng et al. constructed an iteration method to solve the linear matrix equations $AXB = C$ and $A_1XB_1 = C_1,\ A_2XB_2 = C_2$ over symmetric solution X [6,7]; Cai and Chen proposed an iterative algorithm for the least squares bisymmetric solutions of the matrix equations $A_1XB_1 = C_1,\ A_2XB_2 = C_2$ [8]; Liao and Lei found the least-squares solution with the minimum-norm for the matrix equation $(AXB, GXH) = (C, D)$ by making use of the generalized singular value decomposition and the canonical

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correlation decomposition [9]. Dehdi and Hajarian studied the iterative algorithm for the reflexive solutions of generalized coupled Sylvester matrix equations [10]; Wang et al. proposed two iterative algorithm to solve the matrix equations $AXB + CX^TD = E$ [11].

Recently, Ding et al. proposed a new hierarchical identification algorithm which was based on gradient search principle for solving linear matrix equations [12–14,17–20]. In these articles, the problem was discussed by applying the so-called hierarchical identification principle. Using the hierarchical identification method, a linear system is decomposed into some subsystems, and then the unknown parameters of each subsystems are identified successively, e.g., the linear matrix equation $AXB + CXD = F$ can be decomposed into two subsystems [20] $AXB = F_1$ and $CXD = F_2$ to be identified successively, where $F_1 = F - CXD,\ F_2 = F - AXB$. Meanwhile, the convergence rate has some connection with convergence factor μ.

In this paper, we mainly propose a conjugate gradient algorithm (CG) to solve the matrix equations: $A_1XB_1 + C_1X^TD_1 = M_1,\ A_2XB_2 + C_2X^TD_2 = M_2$.

As a matter of convenience, we first introduce some notations. $R^{m\times n}$ is the set of $m\times n$ all real matrices and $R^m = R^{m\times 1}$. $A^T,\ R(A)$ denote the transpose and column space of matrix A, $A\otimes B$ represents the Kronecker product of two matrices A and B, vec $(\cdot)$ is the stretching vector operator, i.e. $vec(A) = (a_1^T, a_2^T, \ldots, a_n^T)^T$ for the matrix $A = (a_1, a_2, \ldots, a_n) \in R^{m\times n}$ $(a_i \in R^m,\ i = 1, 2, \ldots, n)$. At the same time, we define the inner product of two matrices $A, B \in R^{m\times n}$ as $\langle A, B\rangle = trace(B^TA)$, then the matrix norm of A induced by the inner product is Frobenius norm and denoted by $\|A\|$, i.e. $\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{trace(A^TA)}$.

For $A, B \in R^{m\times n}$, if $\langle A, B\rangle = 0$, then we say that A and B are orthogonal. Suppose nonzero matrices $A_1, A_2, \ldots, A_k \in R^{m\times n}$, and $\langle A_i, A_j\rangle = 0$ for all $i \neq j$, we call $\{A_i\}_{i=1}^k$ orthogonal matrix sequences, it is obvious that $A_1, A_2, \ldots, A_k$ is linear independent.

Next, we find the solutions of the following linear matrix equation (1.1), which includes minimum-norm solution, optimal approximation solution to a given matrix and the least-squares solution with the minimum-norm.

$$\begin{cases} A_1XB_1 + C_1X^TD_1 = M_1, \\ A_2XB_2 + C_2X^TD_2 = M_2, \end{cases} \tag{1.1}$$

where, $A_1, A_2 \in R^{p\times m},\ B_1, B_2 \in R^{n\times q},\ C_1, C_2 \in R^{p\times n},\ D_1, D_2 \in R^{m\times q},\ M_1, M_2 \in R^{p\times q}$ are given constant matrices, $X \in R^{m\times n}$ is an unknown matrix to be solved.

2. An iterative method when (1.1) is consistent

In this section, we first give a necessary and sufficient condition of the consistency of the linear matrix equation (1.1), and construct the recursive equation, after that, an iterative algorithm is proposed to solve (1.1).

According to Theorem 4.3.8. and Corollary 4.3.10. in [15], there exists a permutation matrix P_{mn} such that

$$vec(X^T) = P_{mn}\,vec(X).$$

P_{mn} is constructed as follows [15,16], let $P_{mn} \in R^{mn\times mn}$ be a square matrix of order mn, which partitioned into $m\times n$ submatrices such that the ijth submatrix has a 1 in its jith position and zeros elsewhere, i.e.

$$P_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} E_{ij} \otimes E_{ij}^T,$$

where $E_{ij} = e_ie_j^T$ is the elementary matrix of order $m\times n$, and $e_i \in R^m (e_j \in R^n)$ is a column vector with a unity in the ith (jth) position and zeros elsewhere. Then, we have

$$P_{mn}P_{nm} = I_{mn}, \quad P_{mn}^T = P_{mn}^{-1} = P_{nm}, \quad P_{nm}\,vec(X^T) = vec(X),$$

and

$$B \otimes A = P_{mp}^T(A\otimes B)P_{nq},$$

where, $A \in R^{m\times n},\ B \in R^{p\times q}$

Then, (1.1) can be equivalently written as

$$\begin{pmatrix} B_1^T \otimes A_1 + (D_1^T \otimes C_1)P_{mn} \\ B_2^T \otimes A_2 + (D_2^T \otimes C_2)P_{mn} \end{pmatrix} vec(X) = \begin{pmatrix} vec(M_1) \\ vec(M_2) \end{pmatrix} = vec(M_1, M_2). \tag{2.1}$$

The following result is well-known.

Lemma 1. *Let*

$$S := \begin{pmatrix} B_1^T \otimes A_1 + (D_1^T \otimes C_1)P_{mn} \\ B_2^T \otimes A_2 + (D_2^T \otimes C_2)P_{mn} \end{pmatrix} \in R^{2pq\times mn},$$

then, Eqs. (1.1) is consistent if and only if