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TAGE iterative algorithm and nonpolynomial spline basis for the solution of nonlinear singular second order ordinary differential equations

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ABSTRACT

In the present paper, we discuss three point difference method based on nonpolynomial spline basis for the second order ordinary differential equation. Difference schemes are derived for linear and nonlinear case and are used to solve via two parameter alternating group explicit iterative algorithm. The schemes have a fourth and second order of uniform convergence for the choice of the parameters involved in the method. Computational results are presented comparing the two methods in terms of accuracy and execution times. The results indicate the advantage of using parallel implementation of the new method.

1. Introduction

We consider solving the following second order ordinary differential equations (ODE)

$$y'' = F(x, y, y'), \quad y(0) = y_a, \quad y(1) = y_b, \quad 0 \leqslant x \leqslant 1, \tag{1.1}$$

where $F(x,y,y')$ is twice continuously differentiable in the region. The analytical solution of (1.1) for arbitrary choice of the function F cannot be found in general. Some of the numerical methods for the approximate solution and design of parallel algorithm for (1.1) have been provided in the references [1,2]. Many papers have appeared dealing with the continuous approximation of $y(x)$ via spline [3–8]. Pandey and Singh [9] considered second order convergence for a class of two point boundary value problems arising in physiology. Recently researchers have developed a C^∞-differentiable nonpolynomial spline basis that compensates the loss of smoothness inherited by polynomial splines, for the approximate solution of linear two point boundary value problems [10–12].

Numerical computations of nonlinear Eq. (1.1), had been experiencing difficulties in singular cases. Some solutions have been offered in this regard. One such being – Rashidinia et al. [13] considered nonpolynomial spline solution for a class of singular boundary value problems involving nonlinearity in y only, where the computational order of convergence fails to match with the fourth order of theoretical estimates. Hence, in this article there is being made fair attempt to introduce an efficient fourth order accurate method for the solution of nonlinear singular second order ODE and the application of two parameter alternating group explicit (TAGE) and Newton-TAGE method proposed by Sukon and Evans [14] and Evans [15]. Since these methods are explicit in nature and coupled compactly, they are suitable for use on parallel computers. We compare the computational results obtained by the proposed methods with corresponding SOR and Newton-SOR method.

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2. Derivation of the nonpolynomial spline methods

Consider the partitioning $\{x_k : x_k = kh, k = 0(1)N+1, h = 1/(N+1)\}$ and let $S_k(x)$ be the interpolating nonpolynomial which interpolates $y(x)$ at x_k defined as follows

$$S_k(x) = \alpha_k \sin\tau(x-x_k) + \beta_k \cos\tau(x-x_k) + \gamma_k(x-x_k) + \delta_k, \quad k = 0(1)N, \tag{2.1}$$

where α_k, β_k, γ_k, and δ_k are constants and τ is the frequency of the trigonometric functions. In order to calculate the coefficients of Eq. (2.1), we first need to define (see [10]):

$$S''_k(x_k) = M_k, \quad S_k(x_k) = y_k, \quad k = 0(1)N. \tag{2.2}$$

We obtain via algebraic calculations the following expressions

$$\alpha_k = \frac{h^2}{\theta^2 \sin\theta}(M_k\cos\theta - M_{k+1}), \quad \beta_k = -\frac{h^2}{\theta^2}M_k,$$
$$\gamma_k = \frac{1}{h}(y_{k+1}-y_k) + \frac{h}{\theta^2}(M_{k+1}-M_k), \quad \delta_k = y_k + \frac{h^2}{\theta^2}M_k,$$

where $\theta = h\tau$.

From the continuity of the first derivatives of $S_{k-1}(x)$ and $S_k(x)$ at $x = x_k$ for $k = 1(1)N$, i.e. $S'_{k-1}(x_k) = S'_k(x_k), k = 1(1)N$, we obtain

$$y_{k-1} - 2y_k + y_{k+1} - h^2(\alpha M_{k-1} + 2\beta M_k + \alpha M_{k+1}) = 0, \quad k = 1(1)N. \tag{2.3}$$

where

$$\alpha = \frac{1}{\theta\sin\theta} - \frac{1}{\theta^2}, \quad \beta = \frac{1}{\theta^2} - \frac{1}{\theta}\cot\theta.$$

Now, consider the following approximations

$$M_{k\pm1} = F\left(x_{k\pm1}, y_{k\pm1}, \frac{1}{2h}(\pm 3y_{k\pm1} \mp 4y_k \pm y_{k\mp1})\right), \tag{2.4}$$

$$M_k = F\left(x_k, y_k, \frac{1}{2h}(y_{k+1}-y_{k-1}) - \frac{h}{20}(M_{k+1}-M_{k-1})\right). \tag{2.5}$$

The spline scheme 2.3 and 2.4, 2.5 is second and fourth order of uniform convergence for $(\alpha,\beta) = (1/6, 1/3)$ and $(1/12, 5/12)$ respectively. (see [11]).

3. Development of difference equations

We now consider the application of nonpolynomial spline formula (2.3) to the linear singular equation

$$y'' = f(x) - a(x)y' - b(x)y, \quad 0 \leqslant x \leqslant 1. \tag{3.1}$$

From (2.3) and (3.1), after carrying out necessary algebra, we obtain

$$\begin{aligned}(1-h\beta a_k)y_{k-1} &- 2(1-h^2\beta b_k)y_k + (1+h\beta a_k)y_{k+1} - 2h^2\beta f_k \\ &= \frac{h}{2}\left(\frac{h}{5}(h\beta a_k - 10\alpha)b_{k-1} - \frac{h}{10}\beta a_k(a_{k+1}+3a_{k-1}) - \alpha(a_{k+1}-3a_{k-1})\right)y_{k-1} \\ &\quad - \frac{h}{2}\left(\frac{h}{5}(h\beta a_k + 10\alpha)b_{k+1} + \frac{h}{10}\beta a_k(3a_{k+1}+a_{k-1}) + \alpha(3a_{k+1}-a_{k-1})\right)y_{k+1} \\ &\quad + \frac{h}{5}(h\beta(a_{k+1}+a_{k-1})a_k + 10\alpha(a_{k+1}-a_{k-1}))y_k + \frac{h^3}{10}\beta a_k(f_{k+1}-f_{k-1}) + h^2\alpha(f_{k+1}+f_{k-1}).\end{aligned} \tag{3.2}$$

The spline scheme (3.2) is of fourth and second order of convergence However, the scheme fails when the solution is to be determined at $k = \pm 1$. We overcome this difficulty by modifying the method in such a way that the solutions retain the order and accuracy even in the vicinity of the singularity $x = 0$. We use the Taylor's expansions

$$a_{k\pm1} = a_k \pm h a'_k + \frac{h^2}{2}a''_k \pm \frac{h^3}{6}a'''_k + \frac{h^4}{24}a''''_k \pm \frac{h^5}{120}a'''''_k + O(h^6) \text{ etc.} \tag{3.3}$$

Substituting the Taylor's approximations of $a_{k\pm1}$, $b_{k\pm1}$ and $f_{k\pm1}$ into Eq. (3.2) and neglecting higher order terms, we get a linear difference equation of the form

$$sb_k y_{k-1} + 2dg_k y_k + sp_k y_{k+1} = RH_k, \quad k = 1(1)N, \tag{3.3}$$