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An explicit formula for the determinant of a skew-symmetric pentadiagonal Toeplitz matrix

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ABSTRACT

We give the explicit determinant of the general skew-symmetric pentadiagonal Toeplitz matrix in terms of the Tchebechev polynomials.

1. Introduction

From practical point of view, pentadiagonal matrices frequently arise from boundary value problems involving fourth order derivatives and a fast computational formula for the determinants are required to test efficiently the existence of unique solutions of the PDEs. These problem arises in many applications, such as the discretization of partial differential equations (PDEs) in 2D or 3D by finite difference or finite element approximation [1]. Pentadiagonal Toeplitz matrices have received much attention [2] and explicit expressions of their determinants are given in special cases [3].

The main objective of this note is to give an explicit formula for the determinant of the skew-symmetric pentadiagonal Toeplitz matrix

$$\mathbf{P}_m = \begin{pmatrix} 0 & -b & -a & 0 & \cdots & 0 \\ b & 0 & -b & -a & \cdots & \vdots \\ a & b & 0 & -b & \cdots & 0 \\ \vdots & a & b & \ddots & \ddots & -a \\ 0 & \vdots & \ddots & \ddots & \ddots & -b \\ 0 & \cdots & 0 & a & b & 0 \end{pmatrix} \in \mathcal{M}_m(\mathbb{C}), \quad a \neq 0.$$

Formula from which we can derive a simple criteria of the invertibility of the matrix $\mathbf{P}_m$.

Obviously, $\det(\mathbf{P}_m) = 0$ if m is odd, so we can suppose that $m = 2n,\ n \in \mathbb{N}^*$. It is classical [5] that the determinant of a skew-symmetric matrix can always be written as the square of a polynomial in the matrix entries. This polynomial is called the *Pfaffian* of the matrix.

Our formula precise the *Pfaffian* of the matrix $\mathbf{P}_{2n}$:

Theorem 1. *We have*

$$\det(\mathbf{P}_{2n}) = \left[a^n U_n\left(\frac{b}{2a}\right)\right]^2,$$

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where U_n is the Tchebechev polynomial of second kind of degree n [4].

2. Determinant of a tridiagonal block Toeplitz matrix

Let the *block tridiagonal Toeplitz matrix* of the form

$$\mathbf{T}_n = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{C} & \mathbf{A} & \mathbf{B} & \ddots & \vdots \\ \mathbf{0} & \mathbf{C} & \ddots & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \mathbf{B} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{C} & \mathbf{A} \end{pmatrix} \in \mathcal{M}_{Nn}(\mathbb{C}),$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are $N \times N$ matrices and $\mathbf{B}$ is invertible.

Lemma 1. *Let $(\mathbf{M}_i)_{i \geqslant -1}$ the sequence of $N \times N$ matrices defined by the recurrence relation*

$$\mathbf{C}\mathbf{M}_i + \mathbf{A}\mathbf{M}_{i+1} + \mathbf{B}\mathbf{M}_{i+2} = 0 \tag{2.1}$$

for $i = -1, 0, \ldots$, and $\mathbf{M}_{-1} = 0$, $\mathbf{M}_0 = \mathbf{I}_N$. Then

$$\det(\mathbf{T}_n) = (-\det(\mathbf{B}))^{Nn} \det(\mathbf{M}_n)$$

Proof.

Let $\mathbf{X}$ be the block matrix

$$\mathbf{X} = \begin{pmatrix} \mathbf{M}_0 & 0 & 0 & 0 & \cdots & 0 \\ \mathbf{M}_1 & \mathbf{I}_N & 0 & 0 & \cdots & \vdots \\ \mathbf{M}_2 & 0 & \mathbf{I}_N & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \mathbf{M}_{n-2} & \vdots & \vdots & \ddots & \ddots & 0 \\ \mathbf{M}_{n-1} & 0 & 0 & \cdots & 0 & \mathbf{I}_N \end{pmatrix} \in \mathcal{M}_{Nn}(\mathbb{C}).$$

Hence by simple matrices multiplication using the relations (2.1) we get

$$\mathbf{T}_n\mathbf{X} = \begin{pmatrix} \mathbf{0} & \mathbf{Y} \\ -\mathbf{B}\mathbf{M}_n & \mathbf{Z} \end{pmatrix},$$

where $\mathbf{Y}$ is the block triangular submatrix

$$\begin{pmatrix} \mathbf{B} & \mathbf{0}\cdots & \cdots & \mathbf{0} \\ \mathbf{A} & \mathbf{B} & \ddots & \vdots \\ \mathbf{C} & \ddots & \ddots & \mathbf{0} \\ \ddots & \ddots & \ddots & \mathbf{B} \end{pmatrix} \mathcal{M}_{N(n-1)}(\mathbb{C}).$$

Obviously $\det(\mathbf{X}) = 1$ and

$$\det(\mathbf{T}_n\mathbf{X}) = (-1)^{N(n-1)} \det(\mathbf{Y}) \det(-\mathbf{B}\mathbf{M}_n) = (-\det(\mathbf{B}))^{Nn} \det(\mathbf{M}_n).$$

The result follows. □

3. Proof of the main result

We put in this section

$$\mathbf{A} = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -a & 0 \\ -b & -a \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

Hence the matrix $\mathbf{T}_n$ becomes