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Numerical solutions of the modified Burgers' equation by Petrov–Galerkin method

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ABSTRACT

The modified Burgers' equation (MBE) is solved numerically by the Petrov–Galerkin method using a linear hat function as the trial function and a cubic B-spline function as the test function. Product approximation has been used in this method. A linear stability analysis of the scheme shows it to be unconditionally stable. The accuracy of the presented method is demonstrated by two test problems. The numerical results are found in good agreement with the exact solutions.

1. Introduction

The Burgers' equation

$$u_t + uu_x - \nu u_{xx} = 0, \tag{1}$$

where ν is a positive constant and the subscripts x and t denote space and time derivatives respectively was first introduced by Batman [1] and later treated by Burgers' [2] as a mathematical model for turbulence. Since then the equation has found applications in fields as diverse as number theory, gas dynamics, heat conduction, elasticity, etc. The Burgers' equation was solved numerically by various methods such as the finite difference [3], Galerkin [4,5], least squares [6] and collocation methods [7,8] etc. Indeed, the Burgers' equation is a special case of the modified Burgers' equation (MBE) of the form

$$u_t + u^{\mu} u_x - \nu u_{xx} = 0, \tag{2}$$

where μ is a positive constant and ν can be interpreted as viscosity. The MBE has the strong nonlinear aspects of the governing equation in many practical transport problems such as nonlinear waves in medium with low frequency pumping or absorption, ion reflection at quasi perpendicular shocks, turbulence transport, wave processes in thermoelastic medium, transport and dispersion of pollutants in river and sediment transport etc. The initial condition associated with Eq. (2) will be

$$u(a, t_0) = f(x), \quad a \leqslant x \leqslant b, \tag{3}$$

with the boundary conditions

$$u(a,t) = g_1(t) \quad \text{and} \quad u(b,t) = g_2(t), \quad t > t_0. \tag{4}$$

Various numerical methods have been proposed to solve MBE. Ramadan et al. [8] used septic B-spline collocation method for the numerical solution of the MBE with $\mu = 1$ and $\mu = 2$. Ramadan and Danaf [9] also obtained the numerical solution of MBE using quintic B-spline collocation method for the case $\mu = 2$ only. Griewank et al. [10] used non-polynomial spline-functions

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for the numerical treatment of the MBE for the cases $\mu = 1$ and $\mu = 2$. Sachdev et al. [11] obtained large-time asymptotics for periodic solutions of the MBE for the cases $\mu = 2$ and $\mu = 3$. Lattice Boltzmann model for the MBE have been studied by Duan et al. [12] for various initial conditions. In all the numerical techniques mentioned above, the small value of the viscosity v considered was up to 0.001 only. In this work, the Petrov–Galerkin method is developed for the MBE using the linear hat function as the trial function and the cubic B-spline function as the test function. Here the proposed method is shown to represent accurately the various wave packets for the cases $\mu = 2$ and 3 and it can treat small value of v up to 0.0001. The numerical results obtained by this proposed method are compared with the known solutions.

2. The Petrov–Galerkin method

For convenience, the MBE (2) is rewritten as

$$u_t + \frac{1}{\mu+1}(u^{\mu+1})_x - vu_{xx} = 0. \tag{5}$$

The space interval $a \leqslant x \leqslant b$ is discretized with $(N + 1)$ uniform grid points $x_j = a + jh$, where $j = 0,1,2,\ldots,N$, and the grid spacing is given by $h = (b - a)/N$. Let $U_j(t)$ denote the approximation to the exact solution $u(x_j, t)$. Following the Petrov–Galerkin method used in [13], we assume that the approximate solution of Eq. (3) as

$$u_h(x,t) = \sum_{j=0}^{N} U_j(t)\phi_j(x). \tag{6}$$

The product approximation technique [14] is used for treating the nonlinear term in the following manner:

$$u_h^{\mu+1}(x,t) = \sum_{j=0}^{N} U_j^{\mu+1}(t)\phi_j(x), \tag{7}$$

where

$$\phi_j(x) = \begin{cases} 1 + (x - jh)/h, & x \in [x_{j-1}, x_j] \\ 1 - (x - jh)/h, & x \in [x_j, x_{j+1}] \\ 0, & \text{otherwise} \end{cases}$$

The unknown functions $U_j(t)$, $j = 0,1,2,\ldots,N$, are determined from the variational formulation

$$((u_h)_t, \psi_j) + \frac{1}{\mu+1}\left(\left(u_h^{\mu+1}\right)_x, \psi_j\right) - v((u_h)_{xx}, \psi_j) = 0, \tag{8}$$

where ψ_j, $j = 0,1,2,\ldots,N$, are test functions, which are taken to be the cubic B-splines given by

$$\psi_j(x) = \frac{1}{h^3}\begin{cases} (x - x_{j-2})^3, & x \in [x_{j-2}, x_{j-1}] \\ h^3 + 3h^2(x - x_{j-1}) + 3h(x - x_{j-1})^2 - 3(x - x_{j-1})^3, & x \in [x_{j-1}, x_j] \\ h^3 + 3h^2(x_{j+1} - x) + 3h(x_{j+1} - x)^3 - 3(x_{j+1} - x)^3, & x \in [x_j, x_{j+1}] \\ (x_{j+2} - x)^3, & x \in [x_{j+1}, x_{j+2}] \\ 0, & \text{otherwise} \end{cases}$$

and (,) denotes the usual inner product:

$$(f,g) = \int_a^b f(x)g(x)dx.$$

Integrating by parts and using the fact that $\psi(a) = \psi(b) = \psi'(a) = \psi'(b) = 0$, Eq. (8) leads to the formulation

$$((u_h)_t, \psi_j) + \frac{1}{\mu+1}\left(\left(u_h^{\mu+1}\right)_x, \psi_j\right) - v(u_h, (\psi_j)_{xx}) = 0. \tag{9}$$

Performing the integrations on (9) will give the following system of ordinary differential equations (ODEs):

$$\frac{1}{20}\dot{\mathbf{A}} + \frac{1}{4(v+1)h}\mathbf{B} - \frac{v}{h^2}\mathbf{C} = 0, \tag{10}$$

where

$$\begin{aligned} \mathbf{A} &= U_{j-2} + 26U_{j-1} + 66U_j + 26U_{j+1} + U_{j+2}, \\ \mathbf{B} &= -(U_{j-2})^{\mu+1} - 10(U_{j-1})^{\mu+1} + 10(U_{j+1})^{\mu+1} + (U_{j+2})^{\mu+1} \\ \mathbf{C} &= U_{j-2} + 2U_{j-1} - 6U_j + 2U_{j+1} + U_{j+1} \end{aligned}$$