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Wavelet based schemes for linear advection–dispersion equation ☆

K. Sandeep [a,*], Shikha Gaur [b], D. Dutta [c], H.S. Kushwaha [c]

[a] Department of Mechanical Engineering, Institute of Technology, BHU, Varanasi, India
[b] Department of Applied Mathematics, Institute of Technology, BHU, Varanasi, India
[c] Bhabha Atomic Research Centre, Trombay, Mumbai, India

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ABSTRACT

In this paper, two wavelet based adaptive solvers are developed for linear advection–dispersion equation. The localization properties and multilevel structure of the wavelets in the physical space are used for adaptive computational methods for solution of equation which exhibit both smooth and shock-like behaviour. The first framework is based on wavelet-Galerkin and the second is based on multiscale decomposition of finite element method. Coiflet wavelet filter is incorporated in both the methods. The main advantage of both the adaptive methods is the elimination of spurious oscillations at very high Peclet number.

1. Introduction

Many interesting physical systems are characterized by the presence of localized structure or sharp transition, which might occur anywhere in the domain or change their locations in space with time. Popular methods such as finite element, so-called meshless and recently developed wavelet methods, to solve these problems efficiently, use adaptive grid techniques. Adaptive refinement techniques can also be profitably applied in solving partial differential equations useful in many applications, including simulation, animation, computer vision, etc. The currently existing adaptive grid techniques may be roughly classified as either subdivision schemes or basis refinement techniques. The major difference between these approaches is that subdivision schemes solve problems in the physical space by increasing the nodes while basis refinement techniques (including hierarchical basis in finite element method) solve problems in coefficient space. Though both the adaptive grid techniques are well understood, a lot of work has to be done for efficient implementation in complex domain, in particular to reduce computational time.

Wavelet has high potential for fast, hierarchical and locally adaptive algorithms because of their compactly supported refinable basis functions [1–4]. Liandrat and Tchamitchian [5] proposed the first algorithm based on a spatial approximation exploiting the regularity properties of an orthonormal wavelet basis. Beylkin and Keiser [6] used wavelet expansion for adaptively updating numerical solution of nonlinear partial differential equations, which exhibit both smooth and shock-like behaviour. Due to the signal processing base of traditional wavelet, the research in PDE simulations [7,8] was limited to simple domain and boundary conditions. This limitation has been eliminated with the development of the lifting scheme [9] and stable completion [10,11]. By using lifting scheme, Vasilyev and Paolucci [12] developed wavelet collocation method to adapt computational refinements to local demands of the solution. Krysl et al. [13] developed conforming hierarchical adaptive refinement methods where hierarchical refinement treats refinement as the addition of finer level "detail function" to an

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* Corresponding author.
E-mail address: dr.sandeep.kumar@gmail.com (K. Sandeep).

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unchanged set of coarse-level functions. Amaratunga and Sudarshan [14] customized the second-generation wavelets to generate hierarchical basis for finite element method to solve PDEs both hierarchically and adaptively.

Advection–dispersion equation exhibits discontinuity (shocks) after a finite time. Further, the numerical solution shows spurious oscillations when dispersion coefficient is small as compared to the velocity of flow, i.e., at high Peclet number. To capture the singular effects in the solution, the domain would require very fine resolution near singularities. The classical discretization based on uniform grid will be highly uneconomical. The existing numerical techniques use artificial dispersion to overcome the stability problem. A comparison of various methods is discussed by Johnson [15], and Zienkiewicz and Taylor [16]. Main interest of our work is to remove numerical instability by adaptive grid generation (very fine grid at the critical zone) and show the effect of filtering by using wavelets without sacrificing the accuracy of the results. In the paper, two wavelet based methods are presented and the results are compared with some recent finite difference methods.

In the first method, linear advection–dispersion equation is solved by using wavelet-Galerkin method. For calculation of inner product, Newton–cotes method is used which can be replaced by recently developed highly efficient methods [17,18]. The basic idea behind the adaptive solution is simply based on the analysis of wavelet coefficients, which gives information about the region where sharp change starts or ends. At any time step only local matrix reflecting the local changes in the solution, is solved. The method uses efficient data structure of uniform grid and periodic basis function to evaluate the entries of the stiffness matrix.

In the second method, the finest scale finite element solution space is projected onto the scaling and wavelet spaces resulting in the decomposition of high- and low-scale components. Repetition of such a projection results in multi-scale decomposition of the fine scale solution. In the proposed wavelet projection method, the fine scale solution can be obtained by any other numerical method also. Subsequently the properties of the wavelet functions are exploited to eliminate the nodes from the smooth region where the wavelet coefficients will not exceed a preset tolerance. This wavelet-based multi-scale transformation hierarchically filters out the less significant part of the solution, and thus provides an effective framework for the selection of significant part of the solution. In this process, the ‘big’ coefficient matrix at the finest level will be calculated once for complete domain whereas the ‘small’ adaptively compressed coefficient matrix for a priory known localized dynamic zone of high gradient, which will be considerably less expensive to solve, will be used for the solution in every step of the solution. Similar technique is used in the software QUADFLOW [19] using finite volume method.

The paper presents simple, general methods with minimal mathematical framework. The present methods remove a number of implementation headache associated with adaptive grid techniques and is a general technique, independent of domain dimension. These methods have very important and highly practical consequence because they reduce the computational time significantly. Description of the different element of the algorithm in combination with different mathematical comments on the method, are provided. The resulting algorithms, while capturing full generality of methods, are surprisingly simple. A set of concrete, compelling examples based on our implementation is also the contribution of the paper.

The rest of the paper is organized as follows. Advection–dispersion equation is presented in Section 2. Multiscaling using wavelet is briefly discussed in Section 3. In Section 4, Method-I, i.e. wavelet Galerkin method is discussed in detail. Multiscale decomposition of finite element, i.e. Method-II is discussed in Section 5. Results obtained using these two methods and their comparative study is presented in Section 6. Finally, Section 7 contains conclusion.

2. Advection–dispersion equation

Generally pollutant concentration in atmosphere is governed by advection–dispersion equation. The one dimensional advection–dispersion equation can be written as:

$$\frac{\partial C}{\partial t} = -u\frac{\partial C}{\partial x} + D\frac{\partial^2 C}{\partial x^2} - \lambda C, \tag{1}$$

where C is concentration (mg/l), t duration (days), u the flow velocity (m/day), x is the distance along the direction of flow from the upstream boundary of modeled domain (m), D the dispersive coefficient (m^2/day), and λ the decay constant (day^{-1}).

We are considering following boundary and initial condition:

$$\text{at } t \geqslant 0, \quad x = 0, \quad C = C_0(t), \tag{2a}$$

$$\text{at } t \geqslant 0, \quad x = L, \quad \frac{\partial C}{\partial x} = 0, \tag{2b}$$

$$\text{at } t = 0, \quad 0 < x \leqslant L, \quad C = 0, \tag{2c}$$

where C_0 is the concentration of constant magnitude.

Applying weighted residual method in the advection–dispersion Eq. (1), we get

$$\int_0^L w\left(\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} - D\frac{\partial^2 C}{\partial x^2} + \lambda C\right)dx = 0. \tag{3}$$