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Bivariate delta-evolution equations and convolution polynomials: Computing polynomial expansions of solutions

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This Paper is Dedicated to Jose Maria Montesinos Amilibia with Admiration and on Occasion of his 65th Birthday

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ABSTRACT

This paper describes an application of Rota and collaborator's ideas, about the foundation on combinatorial theory, to the computing of solutions of some linear functional partial differential equations. We give a dynamical interpretation of the convolution families of polynomials. Concretely, we interpret them as entries in the matrix representation of the exponentials of certain contractive linear operators in the ring of formal power series. This is the starting point to get symbolic solutions for some functional–partial differential equations. We introduce the bivariate convolution product of convolution families to obtain symbolic solutions for natural extensions of functional-evolution equations related to delta-operators. We put some examples to show how these symbolic methods allow us to get closed formulas for solutions of *genuine* partial differential equations. We create an adequate framework to base theoretically some of the performed constructions and to get some existence and uniqueness results.

1. Introduction

Knuth in [1] introduced the concept of *convolution family* as a sequence

$$F_0(t), F_1(t), \ldots, F_n(t), \ldots$$

such that $F_n(t)$ is a polynomial with degree $(F_n(t)) \leqslant n$, for every $n \in \mathbb{N}$, and satisfying the following convolution condition:

$$F_n(s+t) = F_n(s)F_0(t) + F_{n-1}(s)F_1(t) + \cdots + F_1(s)F_{n-1}(t) + F_0(s)F_n(t) \quad \forall s, t \quad \text{and} \quad n \geqslant 0.$$

As described in [1], many such families are known and they appear frequently in applications. Earlier in [2,3] appeared many families of this kind, all of them associated to some delta-operators. Special families of this type were introduced and studied in [4,5] for different purposes related to combinatorics and with additional conditions related to the evaluation at the integers. The real beginning of the use of these families of polynomials can be dated much earlier as one can see in [2,3] and in the corresponding references.

Knuth pointed out that a simple rule characterizing all non-null convolution families is the following: $(F_n(t))_{n\in\mathbb{N}}$ is a non-null convolution family if and only if there is a formal power series $F(x) = 1 + F_1x + \cdots F_nx^n + \cdots$ such that $F_n(t) = [x^n](F(x))^t$, where the notation $[x^n]$ stands for the coefficient of x^n in the expression

$$(F(x))^t = \sum_{n\geqslant 0} F_n(t)x^n.$$

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Earlier in [2,3] the same result was proved for families associated to delta-operators.

In Section 2, we begin giving a simple dynamical interpretation of the convolution families of polynomials. In Section 3, our main aim is to show how convolution families can be used to find symbolic solutions of some functional equations related to partial linear differential equations. In Section 4, we end the paper stating an adequate ultrametric theoretical framework to base some of the constructions used before and to get some results on the uniqueness of solutions. Our main tool in this last section is the well-known Banach's fixed point theorem, see for example [6].

We characterize the convolution families as the entries of an infinite lower triangular Toeplitz matrix with coefficients in the ring of polynomials $\mathbb{K}[t]$. To do this, we use contractive maps for a natural complete ultrametric d in $\mathbb{K}[[x]]$. In fact, that Toeplitz matrix is the matrix representation of the exponential e^{tL} where L is a contractive $\mathbb{K}[[x]]$-module homomorphism in $(\mathbb{K}[[x]], d)$. The exterior product that we consider in this module is the usual Cauchy product of series. It is easily seen that the action of any such operator L is univocally represented by the Cauchy product by a fixed series $g = \sum_{n\geqslant 1} g_n x^n$. Note that $g(0) = 0$. The family $\{e^{tL}\}_{t\in\mathbb{K}}$ is a one-parameter subgroup of the group of $\mathbb{K}$-linear onto isometries in $(\mathbb{K}[[x]], d)$. The parameter varies through $\mathbb{K}$. After that, we treat what we call *delta-evolution equations* which are obtained by substituting the partial derivative respect to one of the variables by a natural action of a delta-operator in an adequate framework.

We use an interesting paper due to Bacher [7] to give symbolic solutions to some evolution initial value problems in terms of one-parameter subgroups of the Riordan group, see [8–12] for basic references on this group. In our previous papers [13–16] we introduced a different approach and a different notation for this group that we will follow herein.

We usually impose that the independent term is null for many series appearing herein. This is because many operators related to the series are automatically contractive. Then, the complete ultrametric d allows us to get easily many results. Using again some results in [7] we should avoid this condition in some cases. In fact, in the last section we also show how to interpret, in particular, the more general case treated by Bacher in [7] related to the Lie Algebra of the Riordan group.

We also introduce herein a bivariate convolution product of convolution families of polynomials. We do this to find symbolic solutions for some delta-evolution equations involving another delta-operator. We put some examples to show how these symbolic results allow us to find initial value problem solutions for some partial linear differential equations. We introduce and compute solutions for what we call the Rodrigues's-evolution equation induced by a delta-operator. This equation comes from the Rodrigues's formula relating consecutive terms in a polynomial sequence of binomial type, see again [2,3].

We always move in the symbolic world but we put *real* examples. We choose in many examples polynomial initial conditions that help us in the computation. Our natural frameworks are related to some ultrametric rings of formal power series. The advantage of these ultrametrics is that we can recognize easily many contractive functions. Then, the Banach's fixed point Theorem and the iterative procedure described in it are in order. Many of the results we give could be applied, in the realm of analytic functions, to obtain solutions for some partial differential equations when $\mathbb{K}$ is the real or the complex field. We think that some results for functions can be reached from here if one follows the line described in the classical textbook [17]. This is because most of the convolution sequences we use are Sheffer sequences of polynomials in the sense defined in [17]. In fact, they are very closely related to polynomial sequences of binomial type as defined in [2,3]. This paper can be summarized as an application of polynomial sequences of binomial type, or of the Rota and collaborators's Finite Operator Calculus, to compute polynomial expansions of solutions for some families of functional linear partial differential equations associated to delta-operators. See [2,3] for all the related basic definitions.

All along this paper $\mathbb{K}$ represents a field of characteristic zero. $\mathbb{N}$ represents the natural numbers, in this field, including zero. $\mathcal{D}$ denotes the usual derivative operator on power series.

We recommend [18] for information on ultrametrics among many other things. We also recommend [2,3] for information about delta-operators that will be used all along this paper. In particular, both papers describe some ways to construct the convolution family of polynomials associated to a delta-operator. Also in [16] are described recurrence relations to obtain all of them.

We used the classical textbook of Weinberger [19] to get some general knowledge on partial differential equations. We do not use any previous result on this subject. We also used the survey [20] for a general view and historical notes on partial differential equations. We hope that this paper can help along the line on the last paragraph in the introduction in [20]. Literally,

"$\cdots$ computations of solutions of PDE's is the major concern in *scientific computing*.$\cdots$"

2. Convolution sequences as exponentials of contractive module-homomorphisms

Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{K}[[x]]$ the ring of formal power series with coefficients in $\mathbb{K}$.

Consider the complete ultrametric d on $\mathbb{K}[[x]]$ given by

$$d(f,g) = \frac{1}{2^{\omega(f-g)}}, \quad \text{for} \quad f,g \in \mathbb{K}[[x]],$$