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Darboux transformation and Hamiltonian structure for the Jaulent–Miodek hierarchy

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ABSTRACT

Starting from the Jaulent–Miodek spectral problem, we derive the associated hierarchy of nonlinear evolution equations in this paper. It is shown that this hierarchy is completely integrable in the Liouville sense and possesses the Hamiltonian structure. Moreover, by virtue of symbolic computation, two types of Darboux transformations for the whole hierarchy are explicitly constructed, which enables us to find the new soliton-like, shock and anti-shock solutions for the Jaulent–Miodek hierarchy. Figures are presented to discuss the properties of the new soliton-like solutions.

1. Introduction

Soliton theory plays an active role in natural sciences, such as condensed matter physics and fluid dynamics [1–6]. Integrable systems have attracted certain attention in the mathematical and physical communities [7–19], and the relevant theories have been proposed to analyze the integrability of nonlinear evolution equations (NLEEs), such as the Painlevé singularity structure [20–22], Lax pair [23–26], inverse scattering transform (IST) [27–29] and Hamiltonian structure [30–35]. Finite-dimensional Hamiltonian systems have been built [30,31], and progress has been made to the theory of infinite-dimensional Hamiltonian systems, as seen in Refs. [32–35]. Key point in the theory of Hamiltonian systems is to search for a symplectic operator and a sequence of scalar functions [33–35]. Approaches to the Hamiltonian structures of some integrable systems have been proposed, including the trace identity to construct the infinite-dimensional Liouville integrable Hamiltonian systems [31,32].

Recent investigations have been seen on the integrable systems of the Ablowitz–Kaup–Newell–Segur (AKNS) [36–38], Kaup–Newell [39–41] and Jaulent–Miodek (JM) hierarchies [42–44]. One can relate a chosen spectral problem to a hierarchy of NLEEs [45–47]. Spectral problems play a role in developing some methods for solving the initial value problems, e.g., the IST method which deals with the initial data decaying at the infinity [28,29,48]. From a unified point of view, the spectral problems can investigate the algebraic properties of the associated NLEEs [49,50]. For an integrable system with a recursion operator, through the hereditariness and symplectic–cosymplectic factorizations of this operator, an infinite-dimensional Lie algebra and constants of motion in involution can be derived [51,52].

As a special gauge transformation [53,54], a Darboux transformation is an algorithmic procedure to get a series of explicit solutions from a trivial one [55–57]. Darboux transformations of some $(1+1)$-dimensional integrable soliton hierarchies, including the AKNS [55], Gerdjikov–Ivanov [56] and Boiti-Pempinelli-Tu [57] hierarchies, have been constructed.

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In this paper, with symbolic computation[1–6], we will investigate the Darboux transformation and Hamiltonian structure for the JM hierarchy. The outline of the present paper will be as follows: In Section 2, the JM hierarchy will be derived based on the boundary value conditions. In Section 3, we will investigate the Hamiltonian structure and Liouville integrability for the JM hierarchy. In Section 4, we will construct two types of the Darboux transformations for the JM hierarchy. Furthermore, in Section 5, some new soliton-like, shock and anti-shock solutions will be obtained. Finally, our conclusions will be addressed in Section 6.

2. JM hierarchy

We will consider the following JM spectral problem and its auxiliary problem [42–44]:

$$\psi_{xx} = \left[\lambda^2 + \lambda v(x,t) + u(x,t)\right]\psi, \tag{2.1a}$$
$$\psi_t = A\psi + B\psi_x, \tag{2.1b}$$

where $\psi = \psi(x,t)$ is an eigenfunction, $u(x,t)$ and $v(x,t)$ are two potentials, and A and B are the functions which depend on $u(x,t),\ v(x,t)$ and the spectral parameter λ.

The compatibility condition between Eqs. (2.1a) and (2.1b), i.e., $\psi_{xxt} = \psi_{txx}$, yields

$$2A_x + B_{xx} = 0, \tag{2.2}$$
$$u_t + \lambda v_t - 2\lambda^2 B_x - 2uB_x - 2\lambda v B_x - Bu_x - \lambda B v_x - A_{xx} = 0. \tag{2.3}$$

Substituting Eq. (2.2) into (2.3), we have

$$u_t + \lambda v_t = J_1 B + \lambda J_2 B + 2\lambda^2 B_x, \tag{2.4}$$

where $J_1 = -\frac{1}{2}\partial^3 + 2u\partial + u_x$ and $J_2 = 2v\partial + v_x$.

Expanding $B = \sum_{j=0}^{n} b_j \lambda^{n-j}$ where b_j's are all functions which depend on $u(x,\ t)$ and $v(x,\ t)$, and from Eq. (2.4), we have

$$u_t = 2ub_{nx} + b_n u_x - \frac{1}{2}b_{nxxx}, \tag{2.5}$$
$$v_t = 2ub_{n-1x} + 2vb_{nx} + b_{n-1}u_x + b_n v_x - \frac{1}{2}b_{n-1xxx}, \tag{2.6}$$
$$2b_{j+2,x} + J_2 b_{j+1} + J_1 b_j = 0, \quad (j = 0,1,\ldots,n-2), \tag{2.7}$$
$$J_2 b_0 + 2b_{1x} = 0, \tag{2.8}$$
$$2b_{0x} = 0. \tag{2.9}$$

Further, from Eq. (2.9), we choose $b_0|_{(u,v)=(0,0)} = 2^n$, and from Eqs. (2.7) and (2.8), we could prove that

$$b_1|_{(u,v)=(0,0)} = b_2|_{(u,v)=(0,0)} = \cdots = b_n|_{(u,v)=(0,0)} = 0. \tag{2.10}$$

Eqs. (2.5) and (2.6) can be rewritten as

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} J_1 & 0 \\ J_2 & J_1 \end{pmatrix}\begin{pmatrix} b_n \\ b_{n-1} \end{pmatrix}. \tag{2.11}$$

Then, by using Eq. (2.7), we can get

$$\begin{pmatrix} b_{j+2} \\ b_{j+1} \end{pmatrix} = \frac{J}{2}\begin{pmatrix} b_{j+1} \\ b_j \end{pmatrix}, \quad j = 0,1,\ldots,n-2, \tag{2.12}$$

where

$$J = \begin{pmatrix} -\partial^{-1}J_2 & -\partial^{-1}J_1 \\ 2 & 0 \end{pmatrix}. \tag{2.13}$$

Thus, a direct calculation gives

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & J_1 \\ -2\partial & 0 \end{pmatrix}\left(\frac{J}{2}\right)^n \begin{pmatrix} b_1 \\ b_0 \end{pmatrix}, \quad n = 0,1,2,\ldots. \tag{2.14}$$

By using Eq. (2.8), we can get $b_1 = -2^{n-1}v$. Hereby, the soliton hierarchy associated with JM Spectral Problem (2.1a) can be written as

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & J_1 \\ -2\partial & 0 \end{pmatrix} J^n \begin{pmatrix} -\frac{1}{2}v \\ 1 \end{pmatrix}, \quad n = 0,1,2,\ldots. \tag{2.15}$$

The first two nonlinear equations of the isospectral JM hierarchy are, respectively,