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New six-node and seven-node hexagonal finite elements

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ABSTRACT

In this paper, we introduce a six-node and a seven-node hexagonal elements. Both elements are based on the rotated trilinear interpolation in terms of the three-directional coordinates. Optimal *a priori* error estimates are provided for the new elements by decomposing the consistency error into two parts that both can be estimated using the F-E-M test. Some numerical tests are carried out to verify the theoretical analysis. Comparisons to both the previously studied edge-oriented hexagonal elements and the traditional linear triangular elements are also provided to show the efficiency and superiority of the new six-node and seven-node hexagonal elements.

1. Introduction

It is not a simple task to construct a hexagonal finite element. One might consider that the six vertices of a hexagon should determine a bivariate quadratic polynomial which has exactly six degrees of freedom. However, this is not true for most cases. For example, the six vertices of a regular hexagon belong to a same quadratic curve, a circle, see Fig. 1, which causes the resulting system to be singular and thus not uni-solvable. To construct a finite element based on the six vertices or the six edges of a hexagon, function spaces other than the bivariate quadratic polynomial space should be considered. There have been some works based on some non-polynomial spaces, such as rational polynomial spaces [1–11] and piecewise polynomial spaces [12]. However the finite elements studied in those works are exotic, which is not preferred in our study. The purpose of this paper is to find a way to introduce new hexagonal elements based polynomial shape functions.

One of the major theoretical difficulties of nonconforming hexagonal elements comes from the *a priori* error estimates. Since there are totally six vertices and six edges in a hexagon, the consistency error is hard to estimate. A nonconforming hexagonal element with one degree of freedom at each edge was introduced in [13], where the consistency error is automatically bounded since the element is edge-oriented. However it is more natural to consider a vertex-oriented counterpart.

Consistency error plays an important role in the *a priori* error estimates for nonconforming elements. The "patch test" introduced in 1965 by Irons [14,15] was considered to be a necessary and sufficient condition for the convergence of nonconforming element and became popular in applications. However, in early eighties of the last century it was proved that the "patch test" is neither sufficient nor necessary for convergence [16–18]. Meantime, Stummel proposed the "generalized patch test" [19] providing a necessary and sufficient condition for the convergence of nonconforming elements applied to general elliptic boundary value problems. A more practical condition, the "F-E-M-Test" was then proposed by Shi in [20]. However, for the hexagonal elements studied in this paper the F-E-M-Test does not apply directly. We will try to decompose the consistency error into two parts which can both be estimated using the F-E-M test.

The remainder of this paper is organized as follows. In Section 2, we give a necessary introduction to the three-directional coordinates that serves as an important tool to explore the symmetry of a hexagon. Basic notations and lemmas to construct

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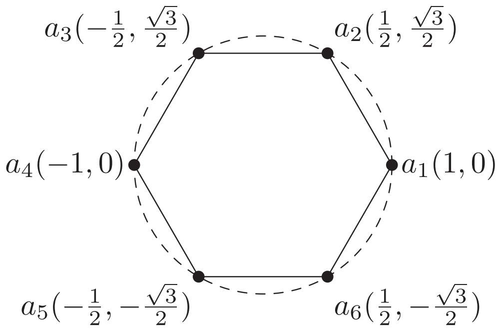

Fig. 1. The six vertices of a regular hexagon lie on a same circle.

hexagonal elements as well as for error estimates are given in Section 3. New hexagonal finite elements and the error estimates are covered in detail in Section 4, and numerical results are reported in Section 5. The paper is concluded in Section 6.

2. Three-directional coordinates for hexagon

Suppose $\mathbf{e}_1$ and $\mathbf{e}_2$ are two independent vectors in the plane, as shown in Fig. 2. Without loss of generality, we assume $\angle(\mathbf{e}_1,\mathbf{e}_2) \in [\frac{\pi}{2},\pi)$. Let $\mathbf{e}_3 = -\mathbf{e}_1 - \mathbf{e}_2$, then the three normal vectors are defined as

$$\mathbf{n}_i := \frac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|^2} \times \mathbf{e}_i, \quad i = 1,2,3. \tag{1}$$

Definition 1 ([21,22]). For any given vector $\mathbf{t}$ in the plane, its three-directional coordinates are defined as $(t_1,t_2,t_3) := (\mathbf{t}\cdot\mathbf{n}_1, \mathbf{t}\cdot\mathbf{n}_2, \mathbf{t}\cdot\mathbf{n}_3)$.

It is easy to verify that $t_1 + t_2 + t_3 \equiv 0$ holds for any vector $\mathbf{t}$. Thus the three-directional coordinates given in Definition 1 are sometimes called homogenous coordinates [23]. In particular, $\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$ can be represented in three-directional coordinates as $(0,-1,1),\quad (1,0,-1),\quad (-1,1,0)$ respectively. Given an origin point, any point P in the plane is belong to a space denoted by $\mathbb{T}^2 := \{(t_1,t_2,t_3) \in \mathbb{T}^3 | t_1 + t_2 + t_3 = 0\}$.

Definition 2. A parallel hexagon of scale ℓ with respect to $\mathbf{e}_1,\ \mathbf{e}_2$ is

$$\mathcal{H}^\ell \equiv \mathcal{H}^\ell(\mathbf{e}_1,\mathbf{e}_2) := \{(t_1,t_2,t_3) \in \mathbb{T}^2 | -\ell < -t_1, t_2, t_3 \leqslant \ell\}. \tag{2}$$

In particular, if $|\mathbf{e}_1| = |\mathbf{e}_2| = |\mathbf{e}_3|$, $\mathcal{H}$ is a regular hexagon which in fact can be obtained by cutting a cube with a plane, as shown in Fig. 3. This is an intuitive reason why we use three coordinates to study hexagons.

Definition 3. Any function $f(\mathbf{t}) : \mathbb{T}^2 \to \mathbb{T}$ is called three-directional periodic with periodicity $\mathbf{s} = (s_1,s_2,s_3) \in \mathbb{T}^2$, if for any point $\mathbf{t} \in \mathbb{T}^2$, there always holds $f(\mathbf{t}) = f(\mathbf{t}+\mathbf{s}),\ \forall \mathbf{s} \in \Lambda_{\mathbf{s}}$, where $\Lambda_{\mathbf{s}} := \mathrm{span}\{(s_1,s_2,s_3),(s_2,s_3,s_1),(s_3,s_1,s_2)\}$. Furthermore, a domain Ω is called a three-directional periodicity domain of $f(\mathbf{t})$, if for any $\mathbf{t} \in \mathbb{T}^2$, there exists and only exists one point $\mathbf{s}^\star \in \Lambda_{\mathbf{s}}$, such that $\mathbf{t} + \mathbf{s}^\star \in \Omega$ and $f(\mathbf{t}) = f(\mathbf{t} + f\mathbf{s}^\star)$.

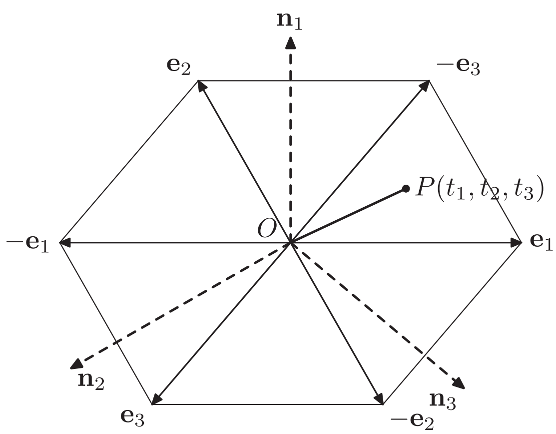

Fig. 2. The three-directional coordinates.