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Global dynamics for a new high-dimensional SIR model with distributed delay

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ABSTRACT

In this paper, a new high-dimensional SIR epidemic model with double epidemic hypothesis and delays is proposed, which is a high-dimensional system of impulsive functional differential equations with time delays. The linear chain trick technique is employed to prove the upper boundedness of solutions of the impulsive delay differential equations and scaling method techniques for inequalities and classification method are used to study the permanence of the high-dimensional system. We also prove that the 'infection-free' periodic solution of the system is globally attractive when $\Re_1 < 1$ and the system is permanent under $\Re_2 > 1$. Moreover, numerical simulation for impulsive and delayed system is presented to illustrate our main conclusions which shows that time delays and pulse vaccination have significant effects on the dynamics behaviors of the model. The feature of the present paper is that the double epidemic hypothesis have different forms of delays to more realistically describe the spread of epidemic though which makes the high-dimensional system more complex.

1. Introduction

The outbreak of infectious diseases had not only caused the loss of billions of lives but also badly damaged the social economy in a short time, which brought much pain to human society. For example, the rage of plague called Antonine brought about the sharp decline of population and deterioration and made the invaders to get a chance to step in, which led to the end of Roman Empire in the 2nd century, B.C [9]. In the "Future plagues: biohazard, disease and pestilence: mankind's battle for survival" [5], Peter Brookesmith described the great danger of infectious diseases to human society, which showed that humans were so vulnerable in the face of infectious diseases. Consequently, it has always been the focus of our research that how to prevent and cure infectious diseases effectively.

As an important biologic model, the SIR infectious disease model has been studied intensively [2–4,16,17,20,22,25]. Among many of the strategies to control infectious diseases, pulse vaccination is an important one and has become a part of our life to prevent the infectious diseases. So it is of great importance and interest to study that in what conditions vaccinated groups would be attacked partly by a given carrier, that is to say, how many people are needed to be vaccinated consistently to avoid the carrier. In recent years many researchers have turned to study pulse vaccination epidemic models, which are formulated as dynamical systems of ordinary differential equations with vaccination in ecology [8,11,15,21,23,28]. Cooke [6] established a famous model for the spread of an infectious disease transmitted by a vector after

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an incubation time, which is well known as "time delay" nowadays and greatly enriches the theoretical research of infectious disease modeling since then. Relying on the theory of delay differential equations, many results have been published [7,14,19,24,26,27,29]. Recently, Meng and Chen [20] studied an impulsive SIR model with discrete delay, and Gao et al. [10] investigated an impulsive SIR model with distributed delay. However, these epidemic models are indeed two-dimensional ODE or IDE with symmetric distributed or discrete delay. In reality, the susceptible can be infected by different epidemics which have different asymmetric distributed or discrete latent delays, and the different types of infectious can compete with the susceptible, which results in a more complicated model with high dimension.

Our goals of this paper are to introduce a new SIR model and to investigate effects of the time delays and pulse vaccination on the dynamical behavior of the model. In the new SIR model, we have the double epidemic hypothesis with both discrete and asymmetric distributed delays, where the asymmetric distributed delay describe spread of one epidemic caused by virus A and the discrete delay describe spread of another epidemic caused by virus B. Also assume both epidemics spread in parallel, and the epidemic caused by virus B which is rather innocuous, protects against epidemic caused by virus A. The model shall be proposed in Section 2. Clearly, both high dimension and asymmetric distributed delay in the SIR model can cause great difficulties in investigating global dynamics for the boundedness, global attractivity and the permanence of the 'infection-extinction' periodic solution, which shall be discussed in Sections 3 and 4, respectively. We then conclude our paper in Section 5 with numerical simulation and discussion.

2. SIR model with both distributed and discrete time delays

Different to the conventional models used previously, in this section, we shall propose a new SIR model, which includes both the distributed and discrete time delays. More precisely, in the model introduced below a population is divided into three classes: the **S**usceptible, **I**nfectious and **R**ecovered. Let $S(t)$ denote the number of members of a population susceptible to the disease, $I_A(t)$ and $I_B(t)$ denote the total population of infectives with virus A and that of infectives with virus B at time t, respectively, and $R(t)$ denote the number of members who have been removed from the possibility of infection through a temporal immunity. $\Delta S(t) = S(t^+) - S(t),\ \Delta I_A(t) = I_A(t^+) - I_A(t),\ \Delta I_B(t) = I_B(t^+) - I_B(t),\ \Delta R(t) = R(t^+) - R(t)$, are used to describe the intervals of time between the pulsed use of controls. Also assume both epidemics spread in parallel, and the epidemic caused by virus B which is rather innocuous, protects against epidemic caused by virus A. Then the model can be read as

$$\begin{cases} \left.\begin{array}{l} S'(t) = \mu - \beta_1 S(t) I_A(t) - \beta_2 e^{-\mu\tau} S(t) I_B(t-\tau) - \mu S(t), \\ I_A'(t) = \beta_1 I_A(t) \int_{-\infty}^{t} F(t-\tau) S(\tau) d\tau - \mu I_A(t) - r_1 I_A(t), \\ I_B'(t) = \beta_2 e^{-\mu\tau} S(t) I_B(t-\tau) - \mu I_B(t) - r_2 I_B(t), \\ R'(t) = r_1 I_A(t) + r_2 I_B(t) - \mu R(t), \end{array}\right\} t \neq nT, \\ \left.\begin{array}{l} \Delta S(t) = -\delta S(t), \\ \Delta I_A(t) = 0, \\ \Delta I_B(t) = 0, \\ \Delta R(t) = \delta S(t), \end{array}\right\} t = nT, \end{cases} \tag{1}$$

where $r_i, \beta_i,\ i = 1,2$ and μ are positive with β_i the contact rate, and r_i the recovery rate from the infected compartment and μ the death rates of susceptibles, infectives and recovered, respectively. T here is the period of the impulsive effect and $n \in N$ with N is the set of all non-negative integers. $0 < \delta < 1$ is the proportion of those vaccinated successfully to all of newborns who become the susceptibles.

Since the population sizes cannot be negative, it is sufficient to consider model (1) with respect to the region $D = \{(S, I_A, I_B, R) \in \mathbb{R}_+^4\}$, where $\mathbb{R}_+^4 = \{x \in \mathbb{R}^4 | x \geqslant 0\}$. Note that the variable R does not appear in the first three equations of model (1), the subsystem concerned can then be written as

$$\begin{cases} \left.\begin{array}{l} S'(t) = \mu - \beta_1 S(t) I_A(t) - \beta_2 e^{-\mu\tau} S(t) I_B(t-\tau) - \mu S(t), \\ I_A'(t) = \beta_1 I_A(t) \int_{-\infty}^{t} F(t-\tau) S(\tau) d\tau - \mu I_A(t) - r_1 I_A(t), \\ I_B'(t) = \beta_2 e^{-\mu\tau} S(t) I_B(t-\tau) - \mu I_B(t) - r_2 I_B(t), \end{array}\right\} t \neq nT, \\ \left.\begin{array}{l} S(t^+) = (1-\delta) S(t), \\ I_A(t^+) = I_A(t), \\ I_B(t^+) = I_B(t), \end{array}\right\} t = nT. \end{cases} \tag{2}$$

We call the discrete delay symmetric due to the fact that both the first and third equations of (2) include $\beta_2 e^{-\mu\tau} S(t) I_B(t-\tau)$. However, the distributed delay is called asymmetric since there is only one term $\beta_1 I_A(t) \int_{-\infty}^{t} F(t-\tau) S(\tau) d\tau$ appearing in the system. Here we choose $F(t) = a e^{-at},\ a > 0$. Then it is easy to verify that $\int_0^{+\infty} F(\tau) d\tau = 1$. Introducing the chain transform $Z(t) = \int_{-\infty}^{t} F(t-\tau) S(\tau) d\tau$, and noticing the fact that

$$\int_{-\infty}^{t} F(t-\tau) d\tau = \lim_{A \to -\infty} \int_{A}^{t} a e^{-a(t-\tau)} d\tau = 1$$