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Existence and exponential stability of a damped wave equation with dynamic boundary conditions and a delay term

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ABSTRACT

In this paper we consider a multi-dimensional wave equation with dynamic boundary conditions related to the Kelvin–Voigt damping and a delay term acting on the boundary. If the weight of the delay term in the feedback is less than the weight of the term without delay or if it is greater under an assumption between the damping factor, and the difference of the two weights, we prove the global existence of the solutions. Under the same assumptions, the exponential stability of the system is proved using an appropriate Lyapunov functional. More precisely, we show that even when the weight of the delay is greater than the weight of the damping in the boundary conditions, the strong damping term still provides exponential stability for the system.

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1. Introduction

In this paper we consider the following linear damped wave equation with dynamic boundary conditions and a delay boundary term:

$$\begin{cases} u_{tt} - \Delta u - \alpha \Delta u_t = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0, \\ u_{tt}(x,t) = -\left(\frac{\partial u}{\partial \nu}(x,t) + \frac{\alpha \partial u_t}{\partial \nu}(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)\right) & x \in \Gamma_1,\ t > 0, \\ u(x,0) = u_0(x) & x \in \Omega, \\ u_t(x,0) = u_1(x) & x \in \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & x \in \Gamma_1,\ t \in (0,\tau), \end{cases} \tag{1}$$

where $u = u(x,t),\ t \geqslant 0,\ x \in \Omega,\ \Delta$ denotes the Laplacian operator with respect to the x variable, Ω is a regular and bounded domain of $\mathbb{R}^N,\ (N \geqslant 1),\ \partial\Omega = \Gamma_0 \cup \Gamma_1,\ mes(\Gamma_0) > 0,\ \overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \varnothing$ and $\frac{\partial}{\partial \nu}$ denotes the unit outer normal derivative, $\alpha,\ \mu_1$ and μ_2 are positive constants. Moreover, $\tau > 0$ represents the time delay and $u_0,\ u_1,\ f_0$ are given functions belonging to suitable spaces that will be precised later.

This type of problems arise (for example) in modeling of longitudinal vibrations in a homogeneous bar in which there are viscous effects. The term Δu_t, indicates that the stress is proportional not only to the strain, but also to the strain rate. See [5]. From the mathematical point of view, these problems do not neglect acceleration terms on the boundary. Such type of

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boundary conditions are usually called *dynamic boundary conditions*. They are not only important from the theoretical point of view but also arise in several physical applications. For instance in one space dimension, problem (1) can modelize the dynamic evolution of a viscoelastic rod that is fixed at one end and has a tip mass attached to its free end. The dynamic boundary conditions represents the Newton's law for the attached mass, (see [4,1,6] for more details). In the two dimension space, as showed in [24] and in the references therein, these boundary conditions arise when we consider the transverse motion of a flexible membrane Ω whose boundary may be affected by the vibrations only in a region. Also some dynamic boundary conditions as in problem (1) appear when we assume that Ω is an exterior domain of $\mathbb{R}^3$ in which homogeneous fluid is at rest except for sound waves. Each point of the boundary is subjected to small normal displacements into the obstacle (see [2] for more details). This type of dynamic boundary conditions are known as acoustic boundary conditions.

In the absence of the delay term (i.e. $\mu_2 = 0$) problem (1) has been investigated by many authors in recent years (see, e.g., [12–15,21,22]).

Among the early results dealing with the *dynamic boundary conditions* are those of Grobbelaar–Van Dalsen [7,8] in which the author has made contributions to this field.

In [7] the author introduced a model which describes the damped longitudinal vibrations of a homogeneous flexible horizontal rod of length L when the end $x = 0$ is rigidly fixed while the other end $x = L$ is free to move with an attached load. This yields to a system of two second order equations of the form

$$\begin{cases} u_{tt} - u_{xx} - u_{txx} = 0, & x \in (0, L),\ t > 0, \\ u(0,t) = u_t(0,t) = 0, & t > 0, \\ u_{tt}(L,t) = -[u_x + u_{tx}](L,t), & t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), & x \in (0, L), \\ u(L,0) = \eta, \quad u_t(L,0) = \mu. \end{cases} \tag{2}$$

By rewriting problem (2) within the framework of the abstract theories of the so-called B-evolution theory, an existence of a unique solution in the strong sense has been shown. An exponential decay result was also proved in [8] for a problem related to (2), which describe the weakly damped vibrations of an extensible beam. See [8] for more details.

Subsequently, Zang and Hu [26], considered the problem

$$u_{tt} - p(u_x)_{xt} - q(u_x)_x = 0, \quad x \in (0,1),\ t > 0$$

with

$$u(0,t) = 0, \quad p(u_x)_t + q(u_x)(1,t) + ku_{tt}(1,t) = 0, \quad t \geqslant 0.$$

By using the Nakao inequality, and under appropriate conditions on p and q, they established both exponential and polynomial decay rates for the energy depending on the form of the terms p and q.

It is clear that in the absence of the delay term and for $\mu_1 = 0$, problem (2) is the one dimensional model of (1). Similarly, and always in the absence of the delay term, Pellicer and Solà-Morales [22] considered the one dimensional problem of (1) as an alternative model for the classical spring-mass damper system, and by using the dominant eigenvalues method, they proved that their system has the classical second order differential equation

$$m_1 u''(t) + d_1 u'(t) + k_1 u(t) = 0,$$

as a limit, where the parameter $m_1,\ d_1$ and k_1 are determined from the values of the spring-mass damper system. Thus, the asymptotic stability of the model has been determined as a consequence of this limit. But they did not obtain any rate of convergence. See also [21,23] for related results.

Recently, the present authors studied in [13,12] a more general situation of (1). They considered problem (1) with $\mu_2 = 0$, a nonlinear damping of the form $g(u_t) = |u_t|^{m-2}u_t$ instead of $\mu_1 u_t$ and a nonlinear source term $f(u) = |u|^{p-2}u_t$ in the right hand side of the first equation of problem (1). A local existence result was obtained by combining the Faedo–Galerkin method with the contraction mapping theorem. Concerning the asymptotic behavior, the authors showed that the solution of such problem is unbounded and grows up exponentially when time goes to infinity if the initial data are large enough and the damping term is nonlinear. The blow up result was shown when the damping is linear. Also, we proved in [12] that under some restrictions on the exponents m and p, we can always find initial data for which the solution is global in time and decay exponentially to zero.

The main difficulty of the problem considered is related to the non ordinary boundary conditions defined on Γ_1. Very little attention has been paid to this type of boundary conditions. We mention only a few particular results in the one dimensional space [16,22,11,17].

The purpose of this paper is to study problem (1), in which a delay term acted in the dynamic boundary conditions. In recent years one very active area of mathematical control theory has been the investigation of the delay effect in the stabilization of hyperbolic systems and many authors have shown that delays can destabilize a system that is asymptotically stable in the absence of delays (see [10] for more details).

In [19], Nicaise and Pignotti examined the wave equation with a linear boundary damping term with a delay. Namely, they looked to the following problem:

$$u_{tt} - \Delta u = 0, \quad x \in \Omega,\ t > 0, \tag{3}$$

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