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# The moment generating function of a bivariate gamma-type distribution

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ABSTRACT

A bivariate gamma-type density function involving a confluent hypergeometric function of two variables is being introduced. The inverse Mellin transform technique is employed in conjunction with the transformation of variable technique to obtain its moment generating function, which is expressed in terms of generalized hypergeometric functions. Its cumulative distribution function is given in closed form as well. Many distributions such as the bivariate Weibull, Rayleigh, half-normal and Maxwell distributions can be obtained as limiting cases of the proposed gamma-type density function. Computable representations of the moment generating functions of these distributions are also provided.



## 1. Introduction

The univariate gamma distribution plays a major role in statistics and is involved in countless applications. The first form of its bivariate extension has been introduced by McKay [13]. An interesting application of this distribution was considered by Clarke [3] in connection with the joint distribution of annual streamflow and areal precipitation. Cherian [2] proposed another form of the bivariate gamma distribution, and Prekopa and Szantai [18] utilized it to study the streamflow of a river.

Kibble [10] and Moran [14] discussed a symmetrical bivariate gamma distribution whose joint characteristic function is given by

$$\frac{1}{\{(1-it_1)(1-it_2)+w^2t_1t_2\}^{\alpha}},\quad \alpha>0, \tag{1.1}$$

where $i=\sqrt{-1}$. Asymmetrical extensions were introduced by Sarmanov [21,22], and another generalization was proposed by Jensen [8] and Smith et al. [23]. Jensen [8] also extended Moran's bivariate gamma distributions.

Recently Nadarajah and Gupta [15] introduced two bivariate gamma distributions based on a characterizing property involving products of gamma and beta random variables. They provided certain representations of their joint densities, product moments, conditional densities and moments. Some of those representations involve special functions such as the complementary incomplete gamma and Whittaker's functions. The Farlie–Gumbel–Morgenstern type bivariate gamma distribution was studied by D'Este [4] and Gupta and Wong [7]. Dussauchoy and Berland [5] introduced a joint distribution in the form of a confluent hypergeometric function of two dependent gamma random variables $X_1$ and $X_2$ with the property that $X_2-\beta X_1$ and $X_1$ are independent. Nakhi and Kalla [17] defined a probability density function involving a generalized $r$-Gauss hypergeometric function and discussed its associated statistical functions. Saxena and Kalla [24] studied a new mixture distribution associated with the Fox–Wright generalized hypergeometric function and discussed some associated

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statistical functions. Nakhi and Kalla [16] discussed further generalizations involving mixture distributions. Provost et al. [19] defined the gamma–Weibull distribution by introducing an additional shape parameter, which also acts somewhat as a location parameter. They provided closed form representations for many associated statistical functions. Saboor and Ahmad [20] defined a bivariate gamma function and its associated density in terms of a confluent hypergeometric function of two variables and discussed some properties of related mathematical functions as well as its probability density function.

In Section 2, we define a bivariate probability density function involving a confluent hypergeometric function of two variables. We provide a representation of its cumulative distribution function. Closed form representations of its moment generating function are derived in the Appendix by making use of an innovative approach that combines the inverse Mellin transform and transformation of variable techniques. The joint moments and marginal pdf's are provided in Section 3. Several particular cases of interest are pointed out in Section 4. The proposed bivariate model is applied to a data set in Section 5 where a parameter estimation technique is being described. The use of the proposed bivariate distribution should lead to modeling improvements in various fields of scientific investigation that rely on Weibull or gamma-type distributions. These include reliability engineering, extreme value theory and failure analysis in the former case, and life testing, industrial engineering (manufacturing times and distribution processes), risk management (probability of ruin) and queuing systems in the latter.

The remainder of this section is devoted to the inverse Mellin transform technique, which is central to the derivation of the moment generating function of the proposed bivariate gamma-type random vector. First, the Mellin transform of a function and its inverse are defined.

If $f(x)$ is a real piecewise smooth function that is defined and single valued almost everywhere for $x > 0$ and such that $\int_0^\infty x^{k-1} \mid f(x) \mid \mathrm{d}x$ converges for some real value $k$, then $M_f(s) = \int_0^\infty x^{s-1} f(x)\mathrm{d}x$ is the Mellin transform of $f(x)$. Whenever $f(x)$ is continuous, the corresponding inverse Mellin transform is

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} M_f(s)\mathrm{d}s, \tag{1.2}$$

which, together with $M_f(s)$, constitute a transform pair. The path of integration in the complex plane is called the Bromwich path. Eq. (1.2) determines $f(x)$ uniquely if the Mellin transform is an analytic function of the complex variable $s$ for $c_1 \leqslant \Re(s) = c \leqslant c_2$ where $c_1$ and $c_2$ are real numbers and $\Re(s)$ denotes the real part of $s$. In the case of a real continuous nonnegative random variable whose density function is $f(x)$, the Mellin transform is its moment of order $(s-1)$ and the inverse Mellin transform yields $f(x)$. When

$$M_f(s) = \frac{\left\{\prod_{j=1}^m \Gamma(b_j + B_j s)\right\}\left\{\prod_{i=1}^n \Gamma(1 - a_i - A_i s)\right\}}{\left\{\prod_{j=m+1}^q \Gamma(1 - b_j - B_j s)\right\}\left\{\prod_{i=n+1}^p \Gamma(a_i + A_i s)\right\}} \equiv h(s), \tag{1.3}$$

where $m, n, p, q$ are nonnegative integers such that $0 \leqslant n \leqslant p,\ 1 \leqslant m \leqslant q,\ A_i,\ i = 1, \ldots, p,\ B_j,\ j = 1, \ldots, q$, are positive numbers and $a_i,\ i = 1, \ldots, p,\ b_j,\ j = 1, \ldots, q$, are complex numbers such that $-A_i(b_j + \nu) \neq B_j(1 - a_i + \lambda)$ for $\nu, \lambda = 0, 1, 2, \ldots,$ $j = 1, \ldots, m$, and $i = 1, \ldots, n$, the $H$-function can be defined as follows in terms of the inverse Mellin transform of $M_f(s)$:

$$f(x) = H_{p,q}^{m,n}\left(x \middle| \begin{matrix}(a_1, A_1), \ldots, (a_p, A_p)\\ (b_1, B_1), \ldots, (b_q, B_q)\end{matrix}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} h(s) x^{-s}\mathrm{d}s, \tag{1.4}$$

where $h(s)$ is as defined in (1.2) and the Bromwich path $(c - i\infty, c + i\infty)$ separates the points $s = -(b_j + \nu)/B_j,\ j = 1, \ldots, m,$ $\nu = 0, 1, 2, \ldots$, which are the poles of $\Gamma(b_j + B_j s),\ j = 1, \ldots, m$, from the points $s = (1 - a_i + \lambda)/A_i,\ i = 1, \ldots, n,\ \lambda = 0, 1, 2, \ldots$, which are the poles of $\Gamma(1 - a_i - A_i s),\ i = 1, \ldots, n$. Thus, one must have

$$\mathcal{M}ax_{1\leqslant j\leqslant m}\Re\{-b_j/B_j\} < c < \mathcal{M}in_{1\leqslant i\leqslant n}\Re\{(1 - a_i)/A_i\}. \tag{1.5}$$

The integral (1.4) converges absolutely when $m + n - p/2 - q/2 > 0$. When $A_i = B_j = 1$ for $i = 1, \ldots, p$ and $j = 1, \ldots, q$, the $H$-function reduces to Meijer's $G$-function, that is,

$$G_{p,q}^{m,n}\left(x \middle| \begin{matrix}a_1, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\right) \equiv H_{p,q}^{m,n}\left(x \middle| \begin{matrix}(a_1, 1), \ldots, (a_p, 1)\\ (b_1, 1), \ldots, (b_q, 1)\end{matrix}\right). \tag{1.6}$$

The following analytic continuation formula can prove useful:

$$G_{p,q}^{m,n}\left(x \middle| \begin{matrix}a_1, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\right) = G_{q,p}^{n,m}\left(\frac{1}{x} \middle| \begin{matrix}1 - b_1, \ldots, 1 - b_q\\ 1 - a_1, \ldots, 1 - a_p\end{matrix}\right). \tag{1.7}$$

For example, when $p = q$, the $G$- and $H$-functions are defined for $0 < x < 1$, and identity (1.7) can be utilized to evaluate these generalized hypergeometric functions for $x > 1$. If, for certain parameter values, an $H$-function remains positive on the entire domain, then whenever the existence conditions are satisfied, a probability density function can be generated by normalizing it. For example, the Weibull, chi-square, half–normal and $F$ distributions can all be expressed in terms of an $H$-function. The main properties of the $H$-function as well as its relationships with other special functions are discussed for instance in Mathai and Saxena [12] and Mathai [11]. These two monographs also contain numerous results of interest in connection with generalized hypergeometric functions and point out certain applications arising in various fields of research such as theoretical physics, operations research, statistical distribution theory and hydrodynamics.