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On the numerical solution of nonlinear Burgers'-type equations using meshless method of lines

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ABSTRACT

In this paper, a meshless method of lines (MMOL) is proposed for the numerical solution of nonlinear Burgers'-type equations. This technique does not require a mesh in the problem domain, and only a set of scattered nodes provided by initial data is required for the solution of the problem using some radial basis functions (RBFs). The scheme is tested for various examples. The results obtained by this method are compared with the exact solutions and some earlier work.

1. Introduction

In this work, we investigate the numerical solution of the equation given by:

$$u_t + \alpha u^\delta u_x - \nu u_{xx} = \beta u(1-u^\delta)(u^\delta - \gamma), \quad a \leqslant x \leqslant b,\ t \geqslant 0, \tag{1}$$

where $u = u(x,t)$, the subscripts t and x represent differentiation with respect to t and x respectively, α, β, γ, δ and ν are constants with $\beta \geqslant 0$, $\delta > 0$, $\gamma \in (0,1)$, and $\nu > 0$, which represents the kinematic viscosity. When $\nu = 1$, Eq. (1) reduces to the generalized Burgers'–Huxley equation which illustrates the interaction between diffusion, convection and reaction [1]:

$$u_t + \alpha u^\delta u_x - u_{xx} = \beta u(1-u^\delta)(u^\delta - \gamma), \quad a \leqslant x \leqslant b, \quad t \geqslant 0. \tag{2}$$

It is to be noted that when $\alpha = 0$ and $\delta = 1$, Eq. (2) reduces to Huxley equation which describes nerve pulse propagation in nerve fibers and wall motion in liquid crystals [2,3]. When $\beta = 0$ and $\delta = 1$, Eq. (2) reduces to Burgers' equation. Burgers' equation is a nonlinear equation which has a convection term, a viscosity term and a time-dependent term. The importance of the study of Burgers' equation is evident from the fact that it has several applications in engineering and environmental sciences. It occurs in the fields of elasticity, gas dynamics, propagation of a shock wave in viscous fluid, heat conduction and turbulence etc. [4–7]. Burgers' equation was first introduced by Bateman [8], who gave its steady solutions [5]. Later, it was studied by Burgers [6] where he used it as a mathematical model of turbulence [9]. Due to the remarkable work by Burgers, this equation is referred to as Burgers' equation.

Many researchers have shown interest in the numerical solution of Burgers'–Huxley equation. A variety of methods have been used to investigate the numerical solution of this equation. Adomian decomposition method (ADM) was introduced by Ismail, Raslan and Rabboh for Burgers'–Huxley and Burgers'–Fisher equation [10]. A numerical solution of the generalized Burgers'–Huxley equation by pseudospectral method and Darvishi's preconditioning was obtained by Javidi [11]. Chebyshev spectral collocation method with preconditioning (CSCMP) was applied by Javidi and Golbabai to find numerical solution of the generalized Burgers'–Huxley equation [12]. Tomasiello used iterative differential quadrature (IDQ) method for

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Burgers'–Huxley equation [13]. Spectral collocation method and Darvishi's preconditioning were applied to solve the generalized Burgers'–Huxley equation by Darvishi et al. [14]. Hon and Mao used RBFs method to solve Burgers' equation [15]. Kutluay solved this equation by using implicit finite difference method (FDM) [16]. In [9] a finite element method (FEM) was introduced for the solution of Burgers' equation. The traditional methods like FDM and FEM use a mesh for space discretization, but mesh generation in higher dimensions is computationally very costly.

In the last couple of decades, RBFs meshless method has attracted many researchers for approximating the solution of partial differential equations (PDEs) in the fields of science and engineering. In 1990, Kansa [17,18] coined the idea of multiquadric (MQ) collocation method using RBFs for the solution of PDEs. Franke and Schaback [19] elaborated the convergence of the method. Flyer and Wright [20] showed that RBFs interpolation is more accurate as compared to other numerical methods even for lower number of nodes and large time-steps. Many researchers have successfully used RBFs to obtain numerical solutions of PDEs in various fields. Elliptic PDEs were solved by Kansa and Hon with MQ RBFs [21]. Haq, Islam and Uddin applied RBFs to find the numerical solution of Korteweg–de Vries–Burgers' equation [22]. Uddin, Haq and Islam used mesh-free collocation method based on RBFs to solve the complex modified Korteweg–de Vries equation [23]. For the detailed applications of RBFs see [17–19,24,25]. In RBFs, FDM is applied for time integration. In this work, we use a recently introduced approach, MMOL [26], to find the numerical solution of nonlinear PDEs.

The method of lines (MOL) is a very powerful numerical method for the solution of time-dependent PDEs. This method was introduced by Schiesser in 1991 [27]. In this method, first the spatial derivatives are approximated by finite differences, which reduces the PDEs to ordinary differential equations (ODEs). Next, the ODEs are integrated in time. Existence of robust ODE solvers makes the MOL a very attractive method. For a detailed study of the method the readers are encouraged to go through the papers [28–31]. In [26] a meshless method of lines was applied to find the numerical solution of KdV equation. More recently, Haq, Bibi, Tirmizi and Usman used MMOL to solve the generalized Kuramoto–Sivashinsky equation [32]. We use MMOL for the numerical solution of Burgers'-type equations. The meshless characteristic of the MMOL gives it an edge over the conventional MOL, which uses a mesh for the domain discretization. In MMOL, RBFs are used to approximate the solution over a set of scattered nodes in the domain [26]. In the present work, $\phi(r) = (r^2 + c^2)^{1/2}$ (MQ), $\phi(r) = \exp(-cr^2)$ (Gaussian (GA)) and $\phi(r) = r^3$ (cubic) are used as RBFs. The time integration is carried out by using fourth order Runge–Kutta (RK4) method.

The rest of the paper is organized as follows: in Sections 2 and 3, the proposed method is given. The test problems are presented in Section 4. The results are summarized in Section 5.

2. RBFs method

Let us choose N distinct nodes ($x_j, j = 1, 2, \dots, N$) in the interval $[a, b]$. In RBFs, a function $u(x,t)$ is approximated by $u^a(x,t)$ as

$$u^a(x,t) = \sum_{j=1}^{N} \lambda_j \phi(r_j) = \mathbf{\Phi}^T(r)\lambda, \tag{3}$$

where $r_j = \|x - x_j\| = \sqrt{(x - x_j)^2}$, λ_j are time dependent unknowns to be determined, and $\phi(r)$, $r \geqslant 0$, is some RBF, also

$$\mathbf{\Phi}(r) = [\phi_1(r), \phi_2(r), \dots, \phi_N(r)]^T$$

and

$$\lambda = [\lambda_1, \lambda_2, \dots, \lambda_N]^T.$$

Let $u^a(x_i, t) = u_i(t)$, then Eq. (3) becomes

$$\mathbf{A}\lambda = \mathbf{u}, \tag{4}$$

where $\mathbf{u} = [u_1(t), u_2(t), \dots, u_N(t)]^T$, and

$$\mathbf{A} = \begin{bmatrix} \mathbf{\Phi}^T(r_1) \\ \mathbf{\Phi}^T(r_2) \\ \vdots \\ \mathbf{\Phi}^T(r_N) \end{bmatrix} = \begin{bmatrix} \phi_1(r_1) & \phi_2(r_1) & \dots & \phi_N(r_1) \\ \phi_1(r_2) & \phi_2(r_2) & \dots & \phi_N(r_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(r_N) & \phi_2(r_N) & \dots & \phi_N(r_N) \end{bmatrix}.$$

From Eqs. (3) and (4) we can write

$$u^a(x,t) = \mathbf{\Phi}^T(r)\mathbf{A}^{-1}\mathbf{u} = \mathbf{N}(r)\mathbf{u}, \tag{5}$$

where $\mathbf{N}(r) = \mathbf{\Phi}^T(r)\mathbf{A}^{-1}$.