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A modified halpern-type iteration algorithm for totally quasi-ϕ-asymptotically nonexpansive mappings with applications

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ABSTRACT

The purpose of this article is to modify the Halpern-type iteration algorithm for total quasi-ϕ-asymptotically nonexpansive mapping to have the strong convergence under a limit condition only in the framework of Banach spaces. The results presented in the paper improve and extend the corresponding results of [X.L. Qin, Y.J. Cho, S.M. Kang, H. Y. Zhou, Convergence of a modified Halpern-type iterative algorithm for quasi-ϕ-nonexpansive mappings, Appl. Math. Lett. 22 (2009) 1051–1055], [Z.M. Wang, Y.F. Su, D.X. Wang, Y.C. Dong, A modified Halpern-type iteration algorithm for a family of hemi-relative nonexpansive mappings and systems of equilibrium problems in Banach spaces, J. Comput. Appl. Math. 235 (2011) 2364–2371], [Y.F. Su, H.K. Xu, X. Zhang, Strong convergence theorems for two countable families of weak relatively nonexpansive mappings and applications, Nonlinear Anal. 73 (2010) 3890–3906], [C. Martinez-Yanes, H.K. Xu, Strong convergence of the CQ method for fixed point iteration processes, Nonlinear Anal. 64 (2006) 2400–2411] and others.

1. Introduction

Throughout this paper we assume that E is a real Banach space with the dual E^*, C is a nonempty closed convex subset of E and $J : E \to 2^{E^*}$ is the *normalized duality mapping* defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \quad x \in E.$$

In the sequel, we use $F(T)$ to denote the set of fixed points of a mapping T, and use $\mathscr{R}$ to denote the set of all real numbers. Recall that a mapping $T : C \to C$ is *nonexpansive*, if $\|Tx - Ty\| \leqslant \|x - y\|$, $\forall x, y \in C$.

One classical way to study nonexpansive mappings is to use contraction to approximate a nonexpansive mapping. More precisely, take $t \in (0,1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1 - t)Tx, \quad \forall x \in C,$$

where $u \in C$ is a fixed point. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in C. It is unclear what is the behavior of x_t as $t \to 0$, even if T has a fixed point. However, for the case of T having a fixed point, Browder [1] proved that if H is a Hilbert space, then x_t converges strongly to a fixed point of T which is the nearest to u.

Motivated by Browder's results, Halpern [2] considered the following explicit iteration:

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$$x_0 \in C, \quad x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad \forall n \geqslant 0, \tag{1.1}$$

where T is nonexpansive. He proved the strong convergence of $\{x_n\}$ to a fixed point of T provided that $\alpha_n = n^{-\theta}$, where $\theta \in (0,1)$.

Recently, many authors improved the result of Halpern [2] and studied the restrictions imposed on the control sequence $\{\alpha_n\}$ in iteration algorithm (1.1). In 2006, Martinez-Yanes and Xu [3] proposed the following modification of the Halpern iteration for a single nonexpansive mapping T in a Hilbert space and proved the following theorem:

Theorem (MYX [3]). *Let H be a real Hilbert space, C be a closed and convex subset of H and $T: C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. If $\{\alpha_n\} \subset (0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$, then the sequence $\{x_n\}$ defined by*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leqslant \|x_n - z\|^2 + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \geqslant 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, n \geqslant 1. \end{cases} \tag{1.2}$$

converges strongly to $P_{F(T)}x_0$.

Very recently Qin et al. [4,5] and Wang et al. [6] improved the result Martinez-Yanes and Xu [3] from Hilbert spaces to Banach spaces for relatively nonexpansive mappings [7,8], quasi-ϕ-nonexpansive mappings and a family of quasi-ϕ-nonexpansive mappings and under suitable conditions some strong convergence theorems are proved.

The purpose of this paper is to consider a hybrid projection algorithm for modifying the iterative process (1.1) to have strong convergence for totally quasi-ϕ-asymptotically nonexpansive mappings which contains relatively nonexpansive mappings, quasi-ϕ-nonexpansive mappings (or hemi-relatively nonexpansive mappings), quasi-ϕ-asymptotically nonexpansive mappings [9] as its special cases. The results presented in the paper extend and improve the corresponding results of Qin et al. [4,5], Wang et al. [6], Martinez-Yanes and Xu [3] and others.

2. Preliminaries

In the sequel, we always use $\phi : E \times E \to \mathscr{R}^+$ to denote the Lyapunov functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, \quad y \in E. \tag{2.1}$$

It is obvious from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \leqslant \phi(x,y) \leqslant (\|x\| + \|y\|)^2, \quad \forall x, \quad y \in E. \tag{2.2}$$

Following Alber [10], the *generalized projection $\Pi_C : E \to C$* is defined by

$$\Pi_C(x) = arg\inf_{y\in C} \phi(y,x), \quad \forall x \in E.$$

Lemma 2.1 [10]. *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Then the following conclusions hold:*

(a) *$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leqslant \phi(x,y)$ for all $x \in C$ and $y \in E$;*
(b) *If $x \in E$ and $z \in C$, then $z = \Pi_C x \iff \langle z - y, Jx - Jz\rangle \geqslant 0, \forall y \in C$;*
(c) *For x, $y \in E$, $\phi(x,y) = 0$ if and only if $x = y$;*

Remark 2.2. If H is a real Hilbert space, then $\phi(x,y) = \|x - y\|^2$ and $\Pi_C = P_C$ (the metric projection of H onto C).

Recall that a point $p \in C$ is said to be an *asymptotic fixed point* of $T : C \to C$ if, there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\|x_n - Tx_n\| \to 0$. Denote the set of all asymptotic fixed points of T by $\widehat{F}(T)$. A point $p \in C$ is said to be a *strong asymptotic fixed point* of T, if there exists a sequence $\{x_n\} \subset C$ such that $x_n \to p$ and $\|x_n - Tx_n\| \to 0$. Denoted the set of all strong asymptotic fixed points of T by $\widetilde{F}(T)$.

Definition 2.3

(1) A mapping $T : C \to C$ is said to be *relatively nonexpansive* [8,11], if $F(T) \neq \emptyset$, $F(T) = \widehat{F}(T)$ and $\phi(p,Tx) \leqslant \phi(p,x)$, $\forall\, x \in C$, $p \in F(T)$.