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Uniform decay in viscoelasticity for kernels with small non-decreasingness zones

Nasser-eddine Tatar

King Fahd University of Petroleum and Minerals, Department of Mathematics and Statistics, Dhahran 31261, Saudi Arabia

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ABSTRACT

We consider a viscoelastic problem with a relaxation function which may be strictly increasing in some sub-intervals. That is we go beyond the zero which was a bound for the derivative of the kernel in the previous works. It is proved that we have different types of decay rates (including the exponential one) provided that the rate and/or the non-decreasingness zone (including the flat zone) is small enough in a certain sense.

1. Introduction

We shall consider the following wave equation with a viscoelastic damping term

$$\begin{cases} u_{tt} = \Delta u - \int_0^t h(t-s)\Delta u(s)ds, & \text{in } \Omega \times \mathbf{R}_+, \\ u = 0, & \text{on } \Gamma \times \mathbf{R}_+, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\Gamma = \partial\Omega$. The functions $u_0(x)$ and $u_1(x)$ are given initial data and the (nonnegative) relaxation function $h(t)$ will be specified later on. Assume that a material, with an instantaneous elasticity and creep characteristics, undergoes two non-simultaneous applied sudden changes in uniform stress, superimposed upon each other. During the period between the two applications, the material responds in some time dependent manner which depends on the magnitude of the stress state. But if we consider the situation that exists at an arbitrarily small interval of time after the sudden application of the second stress, the material not only does experience the instantaneous response to the second stage in surface tractions but also it experiences a continuing time dependent response due to the first applied level of stress. We should note that a purely elastic material would respond only to the total stress level at every instant of time. Thus, this is a more general type of material possessing a characteristic which can be referred to as a memory term. That is, the material response not only does depend on the current state of stress, but also on all past states of stress, and in general sense, the material has a memory keeping all past states of stress. Eq. (1) describes the equation of motion of a viscoelastic body with fading memory (see [2,6,7]).

This kind of problem has been treated by several authors (see [1,3–5,8–32]). Many results on well-posedness and asymptotic behavior have been established. A lot of efforts are devoted to enlarging the class of kernels that lead to some decay of the energy. The specific rate of decay also has attracted the attention of many authors. Briefly, we passed from $h(t) = e^{-\beta t},\ \beta > 0$, to the assumption: $-\xi_1 h(t) \leqslant h'(t) \leqslant -\xi_2 h(t)$, for all $t \geqslant 0$ for some positive constants ξ_1 and ξ_2 together with some conditions on the second derivative. Later these conditions have been relaxed to $h'(t) \leqslant -\xi h(t)$, for all $t \geqslant 0$ and some positive constant ξ to $h'(t) \leqslant -\xi(t)h(t)$, for all $t \geqslant 0$ and some positive function $\xi(t)$ (see [4,12,15]) and then to $h'(t) \leqslant 0$

E-mail address: tatarn@kfupm.edu.sa

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(see [24,30]). The case when $h'(t)$ may be positive has not been well studied (see [14,25]). We refer the reader to the indefinite dissipation case studied by [20] where both the kernel and its derivative may be positive.

In this work we suppose that $h'(t) \leqslant \xi(t)$ where $\xi(t)$ is some non-negative function. The only condition we impose on h is the existence of a function $\mu(t)$ such that

$$h(t-s) \geqslant \mu(t)\int_t^\infty h(\sigma - s)d\sigma, \quad t > 0.$$

This condition is satisfied by a large class of functions including polynomials and exponentials. If ξ satisfies a similar assumption then we prove that the energy decays like $\exp\left(-\int_0^t \mu(s)ds\right)$ or like $\exp(-Ct)$ depending on $\lim_{t\to\infty}\mu(t)$ being equal to zero or different from zero provided that $\xi(t)$ or the non-decreasingness zone is small enough. This seems natural as the system is of fading nature and $\xi(t)$ may be regarded as a perturbation.

The well posedness can be proved using the Faedo Galerkin method (see for instance [3,5]).

Theorem. *Let $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ and h (t) be a nonnegative summable kernel. Then there exists a unique (weak) solution u to problem (1) such that*

$$u \in L^\infty_{loc}(0,\infty; H^2(\Omega)\cap H_0^1(\Omega)), \quad u_t \in L^\infty_{loc}(0,\infty; H_0^1(\Omega)), \quad u_{tt} \in L^\infty_{loc}\left(0,\infty; L^2(\Omega)\right).$$

If $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, then there exists a unique (weak) solution u satisfying

$$u \in C\left([0,\infty); H_0^1(\Omega)\right) \cap C^1\left([0,\infty); L^2(\Omega)\right).$$

The plan of the paper is as follows: in the next section we prepare some material needed to prove our result. We prove an equivalence between the classical energy and the modified energy functional we are going to work together with. Section 3 is devoted to the statement and proof of the uniform decay result.

2. Preliminaries

We define the (classical) energy by

$$E(t) = \frac{1}{2}\int_\Omega \left(|u_t|^2 + |\nabla u|^2\right)dx$$

and the modified energy is

$$\mathcal{E}(t) := \frac{1}{2}\int_\Omega \left\{|u_t|^2 + \left(1 - \int_0^t h(s)ds\right)|\nabla u|^2 + h\Box\nabla u\right\}dx.$$

Using the identity

$$2\int_\Omega \nabla u_t.\int_0^t h(t-s)\nabla u(s)ds\,dx = \int_\Omega (h'\Box\nabla u)dx - h(t)\|\nabla u\|_2^2 - \frac{d}{dt}\left\{\int_\Omega (h\Box\nabla u)dx - \left(\int_0^t h(s)ds\right)\|\nabla u\|_2^2\right\},$$

where

$$h\Box v(t) := \int_0^t h(t-s)|v(t) - v(s)|^2 ds$$

and $\|\cdot\|_2$ denotes the norm in $L^2(\Omega)$, we can see that

$$\mathcal{E}'(t) = \frac{1}{2}\int_\Omega \left((h'\Box\nabla u)dx - h(t)|\nabla u|^2\right)dx. \tag{2}$$

We assume that the kernel is such that

$$1 - \int_0^{+\infty} h(s)ds = 1 - \kappa > 0.$$

Next, we define the standard functionals

$$\Phi_1(t) := \int_\Omega u_t u dx,$$

$$\Phi_2(t) := -\int_\Omega u_t \int_0^t h(t-s)(u(t) - u(s))ds\,dx$$

and the new ones