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Cornish–Fisher expansions about the F-distribution

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ABSTRACT

Cornish and Fisher (1937) [1] gave a fourth order approximation for the percentile of the F-distribution by approximating the cumulants of $Z = 1/2 \log F$. We obtain an alternative formula by approximating the cumulants of F.

Hill and Davis (1968) [2] gave Cornish–Fisher type expansions about any asymptotically normal distribution. We specialize their results to give expansions about the F-distribution.

1. Introduction and summary

The F distribution is one of the most important distributions in statistics. It arises frequently as the null distribution of a test statistic, especially in likelihood-ratio tests, perhaps most notably in the analysis of variance. For a comprehensive account of the theory and applications of the F distribution, we refer the readers to [3].

Because of its prominence in statistical inference, many authors have considered properties of the F distribution. The aim of this note is provide some alternatives to the expansions given by [1] for the F distribution and to illustrate the advantages.

The Cornish–Fisher expansions have received applications in many areas of statistics, see, for example, [4,5]. They also have applications in many applied areas, including risk measures for hedge funds, margin setting of index futures, structural equation models, modified sudden death tests, blind inversion of Wiener systems, GPS positioning accuracy estimation, steady-state simulation analysis, blind separation of post-nonlinear mixtures, cycle time quantile estimation, estimation of the maximum average time to flower, performance of Skart, testing and evaluation, load flow in systems with wind generation, Value-at-Risk portfolio optimization, quantile mechanics, channel capacity in communications theory, economics, financial intermediation and physics. See, for example, [6,10].

The results of this note are organized as follows. In Section 2, we give approximations for the cumulants of the F-distribution. In Section 3 these are used to give Cornish–Fisher expansions for the F-distribution. These differ from those given by [1] as theirs were based on approximating the cumulants of $Z = 1/2 \log F$. In Section 4, we give Cornish–Fisher type expansions about the standardized F-distribution for any random variable Y_n satisfying the Cornish–Fisher assumption on cumulants:

$$EY_n = O(n^{-1/2}), \quad \mathrm{var}(Y_n) = 1 + O(n^{-1}), \quad \kappa_r(Y_n) = O(n^{1-r/2}), \quad r \geqslant 3.$$

Section 5 illustrates its advantage over its analog based on the Z-distribution.

Appendix A specializes a result of [7]: it approximates the first six cumulants of $t(\widehat{w})$ for t any real function when approximations are available for the first six cumulants of a random variable $\widehat{w}$. Appendix B gives some coefficients needed for the Cornish–Fisher type expansions about the standardized F-distribution. The formulas in the appendices were generated using the Maple software.

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2. Approximations for the cumulants of *F*

Suppose $F \sim F_{n_1,n_2}$ the standard F-distribution with n_1 and n_2 degrees of freedom. The Z-distribution is that of $Z = 1/2 \log F$. Set $n = \min(n_1, n_2)$. From [1] we obtain an expansion for the cumulants of Z

$$\kappa_r(Z) \approx \sum_{i=r-1}^{\infty} k_{ri} n^{-i}$$

for $r \geqslant 1$ in terms of $\lambda_1 = n/n_1, \lambda_2 = n/n_2$. The coefficients $\{k_{ji}\}$ needed for the rth order adjustment to Cornish–Fisher expansions for the distribution and percentiles of Z (and hence for F) are:

$$\begin{aligned}
&\text{for } r = 0 : k_{10} = 0, \quad k_{21} = (\lambda_2 + \lambda_1)/2,\\
&\text{for } r = 1 : k_{11} = (\lambda_2 - \lambda_1)/2, \quad k_{32} = \left(\lambda_2^2 - \lambda_1^2\right)/2,\\
&\text{for } r = 2 : k_{22} = \left(\lambda_2^2 + \lambda_1^2\right)/2, \quad k_{43} = \lambda_2^3 + \lambda_1^3,\\
&\text{for } r = 3 : k_{12} = \left(\lambda_2^2 - \lambda_1^2\right)/6, \quad k_{33} = \lambda_2^3 - \lambda_1^3, \quad k_{54} = 3\left(\lambda_2^4 - \lambda_1^4\right),\\
&\text{for } r = 4 : k_{23} = \left(\lambda_2^3 + \lambda_1^3\right)/3, \quad k_{44} = 3(\lambda_2^4 + \lambda_1^4), \quad k_{65} = 12\left(\lambda_2^5 + \lambda_1^5\right).
\end{aligned}$$

Applying Appendix A with $\widehat{w} = Z, t(\widehat{w}) = F = \exp(2\widehat{w}), w = k_{10} = 0$ and $t_r = 2^r$, we obtain

$$\kappa_r(F_{n_1,n_2}) \approx \sum_{i=r-1}^{\infty} a_{ri} n^{-i} \tag{2.1}$$

for $r \geqslant 1$, where the $\{a_{ij}\}$ needed for the rth order expansions are as follows:

$$\begin{aligned}
&\text{for } r = 0 : a_{10} = 1, \quad a_{21} = 2(\lambda_1 + \lambda_2),\\
&\text{for } r = 1 : a_{11} = 2\lambda_2, \quad a_{32} = 8(\lambda_1 + \lambda_2)(\lambda_1 + 2\lambda_2),\\
&\text{for } r = 2 : a_{22} = 4\lambda_2(3\lambda_1 + 4\lambda_2), \quad a_{43} = 48(\lambda_1 + \lambda_2)\left(\lambda_1^2 + 5\lambda_1\lambda_2 + 5\lambda_2^3\right),\\
&\text{for } r = 3 : a_{12} = 4\lambda_2^3,\ a_{33} = 16\lambda_2(6\lambda_1^2 + 21\lambda_1\lambda_2 + 16\lambda_2^2),\ a_{54} = 384(\lambda_1 + \lambda_2)(\lambda_1 + 2\lambda_2)(\lambda_1^2 + 7\lambda_1\lambda_2 + 7\lambda_2^2),\\
&\text{for } r = 4 : a_{23} = 8\lambda_2^2(7\lambda_1 + 11\lambda_2), a_{44} = 96\lambda_2(10\lambda_1^3 + 65\lambda_1^2\lambda_2 + 118\lambda_1\lambda_2^2 + 64\lambda_2^3),\\
&\qquad\qquad a_{65} = 3840(\lambda_1 + \lambda_2)(\lambda_1^4 + 14\lambda_1^3\lambda_2 + 56\lambda_1^2\lambda_2^2 + 84\lambda_1\lambda_2^3 + 42\lambda_2^4).
\end{aligned}$$

3. Simplified Cornish–Fisher expansions for the *F* distribution

The simplified Cornish–Fisher expansions to $O(n^{-5/2})$ for $Y_{nz} = (n/k_{21})^{1/2}(Z - k_{10}), Y_{nF} = (n/a_{21})^{1/2}(F - a_{10})$ are now given by substituting $A_{ri} = k_{21}^{-r/2} k_{ri}, A_{ri} = a_{21}^{-r/2} a_{ri}$, respectively, into the expressions (4.1), (1.5) of [8]. In particular, setting $P_F(x) = P(F_{n_1,n_2} \leqslant x)$, the quantiles of F_{n_1,n_2} are given by

$$P_F^{-1}(\Phi(x)) \approx a_{10} + (a_{21}/n)^{1/2}\left\{x + \sum_{1}^{\infty} n^{-r/2} g_r(x)\right\} \tag{3.1}$$

with a_{10}, a_{21} given by (2.3) and $g_r(x)$ given by (4.1) of [8], and $\Phi(x)$ the standard normal distribution. The advantages of these simplified expansions is that the order of magnitude of each term as a power of $n^{-1/2}$ is clearly expressed, whereas the terms a, b, c, d on page 319 of [1] combine terms of different magnitude. The percentile for Y_{nZ} on page 320 of [1] has in fact been simplified in this way.

Consider the numerical example of [1]:

Example 3.1. When $n_1 = 24$ and $n_2 = 60$ the 5% values of Z and F are $z = 0.26534844$ and $f = \exp(2z) = 1.7001167$. Taking $\Phi(x) = 0.95$, we have $x = 1.6448536270$. The values obtained from the first four correction terms in (3.1) are

Z Method				*F* Method	
Degree	Successive *z*-totals	Equivalent *F*-totals	Successive	Successive	Successive
	$(\widehat{t})$	$(\exp(2t))$	Errors	*F*-totals	Errors
0	0.28091224	1.7538695	+0.05375283	1.5618245	−0.1382922
1	0.26130581	1.6864262	−0.01369047	1.6804350	−0.0196817
2	0.26577432	1.7015654	+0.00144871	1.6968976	−0.0032191
3	0.26529428	1.6999325	−0.00018415	1.6996901	−0.0004266
4	0.26535073	1.7001245	+0.00000779	1.7000630	−0.0000537
correct	0.26534844	1.7001167	0	1.7001167	0