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Wave equation simulation on manifold by unconditional stable schemes ☆

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ABSTRACT

To predict the wave propagation in a given region over time, it is often necessary to find the numerical solution for wave equation. With the techniques of discrete differential calculus, we propose two unconditional stable numerical schemes for simulation wave equation on the space manifold and the time. The analysis of their stability and error is also accomplished.

1. Introduction

To predict the wave propagation in spacetime manifold, it is often necessary to find the numerical solution for wave equation. The finite difference time domain method is widely used to solve this equation numerically [1–8]. However, this algorithm is limited to compute the wave equation in the region of flat spacetime. The wave equation in manifold can be written as an exterior differential system (EDS). Discrete exterior calculus (DEC) can give a kind of discrete version for the exterior differential forms, Hodge star, exterior derivative, and Laplace operator [9–17]. Therefore, it is a right framework in which to develop a numerical computational method for EDS, and therefore for wave equation written by EDS on manifold.

If using conditional stable computational scheme for wave equation and other hyperbolic differential equation, refining the space mesh to improve the accuracy of the solution will significantly increase the amount of computation, since the time step should also be reduced to satisfy the stable condition. In this paper, we propose two unconditional stable schemes for the numerical solution of wave equation in the space manifold and the time using DEC, namely the implicit and semi-implicit DEC schemes. The analysis of their stability and error is also accomplished.

2. Preliminaries

In its simplest form, the wave equation refers to a scalar function u on the space manifold and the time that satisfies:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u, \tag{1}$$

where Δ is the Laplace's operator and c is the propagation speed of the wave.

The 2D or 3D space manifold can be approximated by triangles or tetrahedrons, and the time by line segments. Suppose each simplex contains its circumcenter. The circumcentric dual cell $D(\sigma_0)$ of simplex σ_0 is

$$D(\sigma_0) := \bigcup_{\sigma_0 \in \sigma_1 \in \cdots \in \sigma_r} \text{Int}(c(\sigma_0)c(\sigma_1)\cdots c(\sigma_r)),$$

where σ_i is all the simplices which contains $\sigma_0, \ldots, \sigma_{i-1}$, and $c(\sigma_i)$ is the circumcenter of σ_i.

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A discrete differential k-form, $k \in \mathbb{Z}$, is the evaluation of the differential k-form on all k-simplices. Dual forms, i.e., forms evaluated on the dual cell. In DEC, the exterior derivative d is approximated as the transpose of the incidence matrix of k-cells on $k+1$-cells, the approximated Hodge star $*$ scales the cells by the volumes of the corresponding dual and primal cells, and the Laplace operator is approximated as

$$\Delta \approx *^{-1} d^T * + d^T * d.$$

Using the central time difference and discrete exterior calculus, we obtain a explicit scheme

$$\delta_t^2 u_i^{n-1} = c^2 d^T * d u_i^n, \tag{2}$$

where i is any vertice in mesh, $n\Delta t$ denotes the coordinate of the time, and

$$\delta_t u^n := \frac{1}{\Delta t}(u^{n+1} - u^n).$$

3. Implicit DEC scheme

Using the backward time difference and discrete exterior calculus, we obtain an implicit scheme

$$\delta_t^2 u_i^{n-1} = c^2 d^T * d u_i^{n+1},$$

For some situations, a source having azimuthal symmetry about its axis is considered. In this case, we only need to consider 2D triangular discrete manifold as the space. We can also use a similar approach to the wave equation in 3D spacial manifold and the time. At first, we define some values on mesh. Take Fig. 1 as an example for a part of 2D mesh, in which $0, \ldots, C$ are vertices, 1, 2, 3 are the circumcenters of triangles, a, b, c are the circumcenters of edges. Denote l_{ij} as the length of line segment (i,j) and A_{ijkl} as the area of quadrangle (i,j,k,l).

Define

$$l_{12} := l_{1b} + l_{2b}, l_{23} := l_{2c} + l_{3c}, l_{31} := l_{3a} + l_{1a},$$

and

$$P_{123} := A_{01ab} + A_{02bc} + A_{03ac}.$$

The implicit scheme on the part of mesh(see Fig. 1) is as follows:

$$u_0^{n+1} = 2u_0^n - u_0^{n-1} + \frac{(c\Delta t)^2}{P_{123}}\left[\frac{l_{13}}{l_{A0}}(u_A^{n+1} - u_0^{n+1}) + \frac{l_{12}}{l_{B0}}(u_B^{n+1} - u_0^{n+1}) + \frac{l_{23}}{l_{C0}}(u_C^{n+1} - u_0^{n+1})\right]. \tag{3}$$

Now, we analyze the stability for scheme (3). Suppose

$$u_0^{n+1} = \xi u_0^n \quad u_0^{n-1} = \frac{1}{\xi}u_0^n \quad u_i^{n+1} = \xi\cos(kl_{0i})u_0^n, \tag{4}$$

where ξ is the growth factor of time, and k is the spatial frequency spectrum. Substituting (4) into scheme (3), we obtain

$$\xi u_0^n = 2u_0^n - \frac{1}{\xi}u_0^n + \frac{(c\Delta t)^2}{P_{123}}\left[\frac{l_{13}}{l_{A0}}\xi(\cos(kl_{0A}) - 1) + \frac{l_{12}}{l_{B0}}\xi(\cos(kl_{0B}) - 1) + \frac{l_{23}}{l_{C0}}\xi(\cos(kl_{0C}) - 1)\right]u_0^n.$$

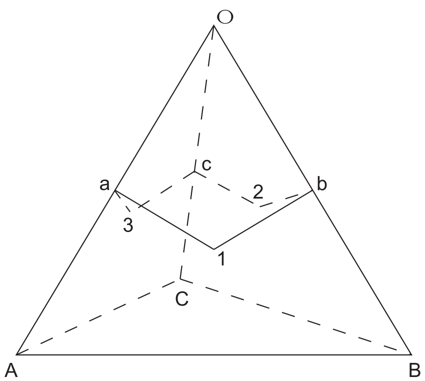

Fig. 1. A part of 2D mesh.