Applied Mathematics and Computation 218 (2011) 520–531

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# A study of Galerkin method for the heat convection equations

Polina Vinogradova *, Anatoli Zarubin

*Department of Natural Sciences, Far Eastern State Transport University, 680021, Khabarovsk, Serisheva 47, Russia*

ARTICLE INFO

*Keywords:*
Approximate solution
Error estimate
Galerkin method
Heat convection equation
Orthogonal projection

ABSTRACT

This article investigates the Galerkin method for an initial boundary value problem for heat convection equations. The new error estimates for the approximate solutions and their derivatives in strong norm are obtained.



## 1. Introduction

The large number of works is devoted to the study of various convection problems. It is possible to recognize two important directions of the study of convection phenomena. The first one is experimental and theoretical study of the convective stability. In detail these questions are considered, for example, in the monograph [1]. The other important direction is the numerical modeling of convection processes (see, for example, [2–5]). It allows to calculate the modes of convection at various meanings of Rayleigh, Reynolds numbers and at other parameters of the model. It is known that the main theoretical basis of numerical methods is the proof of convergence of the approximate solution to the exact one of the corresponding differential problem. The order of the convergence speed of approximate solutions of a nonlinear problem much depends on a kind of the nonlinear terms. It is often difficult to establish the convergence. In this case the basic information on the convergence of the computing procedure is found out by numerical experiments.

In the present paper we study the Galerkin method for the approximate solving of an initial boundary value problem for a non-stationary quasi-linear system which describes the motion of the non-uniformly heated viscous incompressible fluid. The convergence of the Galerkin approximations in a strong norm is established, and also the asymptotic error estimates for the solutions and their derivatives in the uniform norm are obtained.

## 2. Statement of the problem and auxiliary assertions

Let $\Omega$ be a bounded domain in $R^2$with the smooth boundary $\partial\Omega$, $Q = \Omega \times (0,T)$, $S = \partial\Omega \times (0,T]$, where $T < \infty$.

The initial boundary value problem for the heat convection in Boussinesq approximation is formulated in the following way ([1, 6, 7]): we seek a vector-function $u(x,t):\Omega \times [0,T] \to R^2$ and scalar functions $p(x,t)$, $\theta(x,t):\Omega \times [0,T] \to R$ such that

$$\frac{\partial u}{\partial t} - \nu\Delta u + \rho_0^{-1}\nabla p + (u\cdot\nabla)u - g\beta k_3\theta = f(x,t), \quad (x,t) \in Q, \tag{1}$$

$$\frac{\partial \theta}{\partial t} - \kappa\Delta\theta + u\cdot\nabla\theta = \varphi(x,t), \quad (x,t) \in Q, \tag{2}$$

$$\operatorname{div} u(x,t) = 0, \quad (x,t) \in Q, \tag{3}$$

$$u(x,t) = 0, \quad \theta(x,t) = 0, \quad (x,t) \in S, \tag{4}$$

$$u(x,0) = 0, \quad \theta(x,0) = 0, \quad x \in \overline{\Omega}, \tag{5}$$

* Corresponding author.
*E-mail addresses:* vpolina17@hotmail.com (P. Vinogradova), zarubin21@rambler.ru (A. Zarubin).

0096-3003/$ - see front matter 
doi:10.1016/j.amc.2011.05.095

where $\nu$, $\kappa$, $\rho_0$, $\beta > 0$ are some phisical constants; $k_3$ is a vertically directed up unit vector.

Let $L_p(\Omega)$, $1 < p < +\infty$, (respectively $L_\infty(\Omega)$) be a space of real functions are absolutely integrable on $\Omega$ with the power of $p$ according to Lebesque measure $dx = dx_1dx_2$ (respectively essentially bounded). These spaces with the norms

$$\|u\|_{L_p(\Omega)} = \left(\int_\Omega |u(x)|^p dx\right)^{1/p}$$

respectively

$$\|u\|_{L_\infty(\Omega)} = \operatorname{ess}\ \sup_\Omega |u(x)|$$

are Banach spaces. The space $L_p(Q)$ is defined similarly. The Sobolev space $W_p^m(\Omega)$ is a space of functions from $L_p(\Omega)$ whose all generalized partial derivatives up to order $m$ inclusively belong to $L_p(\Omega)$ ($m$ is a nonnegative integer). It is a Banach space with the norm

$$\|u\|_{W_p^m(\Omega)} = \left(\sum_{|j|\leqslant m} \|D^j u\|^p_{L_p(\Omega)}\right)^{1/p}.$$

The space $W_p^{2m,m}(Q)$ (see [8]) with $m$being a nonnegative integer is a Banach space of functions from $L_p(Q)$, which have generalized derivatives $D_t^r D_x^s$ with arbitrary nonnegative integers $r$ and $s$ satisfying inequality $2r + s \leqslant 2m$. The norm in $W_p^{2m,m}(Q)$ is defined as

$$\|u\|_{W_p^{2m,m}(Q)} = \sum_{j=0}^{2m} \sum_{2r+s=j} \|D_t^r D_x^s u\|_{L_p(Q)}.$$

We put

$$W_p^{1,0}(Q) = \{u \in L_p(Q) : D_x u \in L_p(Q)\},$$
$$\overset{\circ}{W}{}_2^1(\Omega) = \left\{u \in W_2^1(\Omega) : u = 0 \text{ on } \partial\Omega \text{ in the sense of traces}\right\}.$$

The symbol $\overset{\circ}{W}{}_2^{2,1}(Q)$ denotes the set of functions belonging $W_2^{2,1}(Q)$ satisfying zero initial conditions and vanishing on $S$.

We shall deal with two-dimensional vector-functions, each component of which belongs to one of the above defined spaces. We denote $[L_p(\Omega)]^2 = L_p(\Omega) \times L_p(\Omega)$, $[L_p(Q)]^2 = L_p(Q) \times L_p(Q)$, etc. The norm, for example, in $[L_p(\Omega)]^2$ ($p > 2$) we shall denote by $[\cdot]_{L_p(\Omega)}$. Let us denote similarly the norms in the spaces $\left[W_2^2(\Omega)\right]^2, [L_p(Q)]^2, \left[W_2^{2,1}(Q)\right]^2$.

The norm in $L_2(\Omega)$ and in $[L_2(\Omega)]^2$ will be denoted by $\|\cdot\|$ and $[\cdot]$, respectively. The inner product in $L_2(\Omega)$ and in $[L_2(\Omega)]^2$ will be denoted by $(\cdot,\cdot)$.

Let $J(\Omega)$ be a set of solenoidal infinitely differentiable and finite on $\Omega$ vectors $v(x) = (v_1(x), v_2(x))$, $\overset{\circ}{J}(\Omega)$ be the closure with respect to the norm of space $\left[W_2^1(\Omega)\right]^2$.The elements of $\overset{\circ}{J}(Q)$ are the vectors $v(x,t)$ that belong to $\overset{\circ}{J}(\Omega)$ for almost all $t$. Let $P_J$ be the orthogonal projection in $[L_2(\Omega)]^2$ onto $\overset{\circ}{J}(\Omega)$.

Using the operator $P_J$, the problem (1)–(5) can be written as

$$\frac{\partial u}{\partial t} - \nu P_J \Delta u + P_J((u\cdot\nabla)u) - g\beta P_J(k_3\theta) = P_J f(x,t), \quad (x,t) \in Q, \tag{6}$$

$$\frac{\partial \theta}{\partial t} - \kappa\Delta\theta + u\cdot\nabla\theta = \varphi(x,t), \quad (x,t) \in Q, \tag{7}$$

$$u(x,t) = 0, \theta(x,t) = 0, \quad (x,t) \in S, \tag{8}$$

$$u(x,0) = 0, \theta(x,0) = 0, \quad x \in \overline{\Omega}. \tag{9}$$

We consider the spectral problems

$$-\nu P_J \Delta e = \lambda e, \quad e \in \overset{\circ}{J}(\Omega),$$
$$e(x) = 0, \quad x \in \partial\Omega$$

and

$$-\kappa\Delta m = \mu m,$$
$$m(x) = 0, \quad x \in \partial\Omega.$$

By $\lambda_i$ we denote an eigenvalue, corresponding to the eigenvector $e_i(x)$, by $\mu_i$ we denote an eigenvalue, corresponding to the eigenvector $m_i(x)$. The existence and the completeness of the eigenfunctions $e_i(x) \in \left[W_2^2(\Omega)\right]^2 \cap \overset{\circ}{J}(\Omega), m_i(x) \in W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$ in the spaces $[L_2(\Omega)]^2$ and $L_2(\Omega)$ are proved in [9, 10].

Let $P_{n1}$ be the orthogonal projection in $[L_2(\Omega)]^2$ onto the linear span of the vector-functions $\{e_i(x)\}_{i=1}^n, P_{n2}$ be the orthogonal projection in $L_2(\Omega)$ onto the linear span of the functions $\{m_i(x)\}_{i=1}^n$.

The approximate solutions for the problem(6)–(9) are defined as