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A pathway from Bayesian statistical analysis to superstatistics

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ABSTRACT

Superstatistics and Tsallis statistics in statistical mechanics are given an interpretation in terms of Bayesian statistical analysis. Subsequently superstatistics is extended by replacing each component of the conditional and marginal densities by Mathai's pathway model and further both components are replaced by Mathai's pathway models. This produces a wide class of mathematically and statistically interesting functions for prospective applications in statistical physics. It is pointed out that the final integral is a particular case of a general class of integrals introduced by the authors earlier. Those integrals are also connected to Krätzel integrals in applied analysis, inverse Gaussian densities in stochastic processes, reaction rate integrals in the theory of nuclear astrophysics and Tsallis statistics in nonextensive statistical mechanics. The final results are obtained in terms of Fox's *H*-function. Matrix variate analogue of one significant specific case is also pointed out.

1. Introduction

Nonequilibrium systems in various areas of science [6,19] exhibit spatio-temporal dynamics that is inhomogeneous and can be described by a superposition of several statistics on different scales, in short called a superstatistics approach in the physics literature [2–5]. Essential for this superstatistics approach is the existence of sufficient time scale separation between two relevant dynamics within the nonequilibrium system. There must be some intensive parameter that fluctuates on a larger time scale than the typical relaxation time of the local dynamics. In a statistical thermodynamic setting such a parameter can be interpreted as a local inverse temperature of the system, but much broader interpretations are possible depending on the nonequilibrium system under consideration. The stationary distributions of superstatistical systems, obtained by averaging over all relevant fluctuating parameters exhibit non-Gaussian behaviour with fat tails, which can be a power law, or a stretched exponential or other functional forms like Mittag–Leffler function or Fox *H*-function. The relevant superstatistical parameter can be an effective parameter in a stochastic differential equation or a local variance parameter extracted from a time series. Currently, in the physical literature, superstatistics [1,3,5] and Tsallis statistics [15–18] are the two preferred models to describe nonequilibrium systems in a sense of generalizing the well-established Boltzmann–Gibbs statistical mechanics.

In this paper we shall develop a Bayesian approach to superstatistics and Tsallis statistics in terms of the recently introduced pathway model [8]. It will be shown that Mathai's pathway model comprises the distributions of superstatistical systems as well as Tsallis statistics in a unified manner.

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2. Bayesian statistical analysis

Let us start with the standard Bayesian analysis problem. Consider a positive real scalar random variable x and a parameter θ. Let the conditional density of x at a given value of θ be denoted by $f(x|\theta)$ and the marginal density of θ by $g(\theta)$ respectively. For simplicity, let us take both $f(x|\theta)$ and $g(\theta)$ in the generalized gamma family of functions. Let

$$f(x|\theta) = \frac{\delta\theta^{\gamma}}{\Gamma\left(\frac{\gamma}{\delta}\right)} x^{\gamma-1} e^{-\theta^{\delta}x^{\delta}}, \tag{1}$$

for $\theta > 0$, $\delta > 0$, $x \geqslant 0$, $\gamma > 0$ and $f(x|\theta) = 0$ elsewhere. Let

$$g(\theta) = \frac{\delta b^{\rho/\delta}}{\Gamma\left(\frac{\rho}{\delta}\right)} \theta^{\rho-1} e^{-b\theta^{\delta}}, \tag{2}$$

for $b > 0$, $\theta > 0$, $\delta > 0$, $\rho > 0$ and $g(\theta) = 0$ elsewhere. Then the unconditional density of x is given by the following:

$$\begin{aligned} f_x(x) &= \int_{\theta=0}^{\infty} f(x|\theta)g(\theta)\mathrm{d}\theta = \frac{\delta^2 b^{\rho/\delta} x^{\gamma-1}}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\rho}{\delta}\right)} \int_0^{\infty} \theta^{\gamma+\rho-1} e^{-\theta^{\delta}(b+x^{\delta})}\mathrm{d}\theta = \frac{\delta b^{\rho/\delta}\Gamma\left(\frac{\gamma+\rho}{\delta}\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\rho}{\delta}\right)} \frac{x^{\gamma-1}}{(b+x^{\delta})^{\frac{\gamma+\rho}{\delta}}}, \quad b + x^{\delta} > 0 \\ &= \frac{\delta\Gamma\left(\frac{\gamma+\rho}{\delta}\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\rho}{\delta}\right) b^{\gamma/\rho}} x^{\gamma-1} \left(1 + \frac{x^{\delta}}{b}\right)^{-\left(\frac{\gamma+\rho}{\delta}\right)}, \end{aligned} \tag{3}$$

for $x \geqslant 0$, $\gamma > 0$, $\rho > 0$, $\delta > 0$, $b > 0$.

Theorem 1. *Let the conditional density of x given θ, that is, $f(x|\theta)$, and the marginal density of θ be as in (1) and (2) respectively. Then the unconditional density of x, denoted by $f_x(x)$, is given by (3).*

In a physical system the parameter θ in (1) may represent temperature so that the density $f(x|\theta)$ may represent the production of the item x at a fixed temperature θ or at a given value of θ. Then the marginal density of θ in (2) may represent the temperature distribution. What is the distribution of the production of x over all temperature variations or averaged over the density of θ? This is the unconditional density of x given in (3) for the specific densities in (1) and (2). Since a density such as the one in (2) is superimposed over the density such as the one in (1), the resulting density in (3) is called superstatistics. Various interpretations of x and θ in different physical systems may be seen in the original paper on superstatistics [1]. Eq. (3) for $\delta = 1,\ \gamma = 1,\ b = \frac{1}{q-1},\ q > 1$ and $\frac{\gamma+\rho}{\delta} = \frac{1}{q-1}$ is Tsallis statistics of non-extensive statistical mechanics, see [15]. The Bayes' density of θ is

$$g_1(\theta|x) = \frac{f(x|\theta)g(\theta)}{f_x(x)}, \tag{4}$$

which is available from (1)–(3). The Bayes' estimate of θ is the conditional expectation of θ at given value of x, denoted by $E(\theta|x)$, where E denotes statistical expectation. This is available from the conditional density of θ, given x, in (4). Thus, Bayes' procedure, superstatistics and Tsallis statistics are all connected together as shown in (1)–(4).

3. Extension of Bayes' procedure

We can extend the discussion in (1)–(4) by using the pathway model of Mathai [8]. For example, let us replace $g(\theta)$ in (2) by a pathway density, namely,

$$P_1(\theta) = \frac{\delta[b(\beta-1)]^{\rho/\delta}\Gamma\left(\frac{\eta_1}{\beta-1}\right)}{\Gamma\left(\frac{\rho}{\delta}\right)\Gamma\left(\frac{\eta_1}{\beta-1} - \frac{\rho}{\delta}\right)} \theta^{\rho-1}[1 + b(\beta-1)\theta^{\delta}]^{-\frac{\eta_1}{\beta-1}}, \tag{5}$$

for $\beta > 1,\ \delta > 0,\ \eta_1 > 0,\ b > 0,\ \rho > 0,\ \theta > 0,\ \frac{\eta_1}{\beta-1} - \frac{\rho}{\delta} > 0$ and $P_1(\theta) = 0$ elsewhere. When $\theta \to 1_+$ in (5) the model in (5) goes to $g(\theta)$ in (2) with b replaced by $b\eta_1$. The model in (5) consists of three different functional forms. For $\beta < 1$ we may write

$$[1 + b(\beta-1)\theta^{\delta}]^{-\frac{\eta_1}{\beta-1}} = [1 - b(1-\beta)\theta^{\delta}]^{\frac{\eta_1}{1-\beta}}.$$

The right side remains positive in the finite range $1 - b(1-\beta)\theta^{\delta} > 0$ or $0 < \theta < [b(1-\beta)]^{-\frac{1}{\delta}}$. Then for $\beta < 1$ the density in (5) changes to the form

$$P_2(\theta) = \frac{\delta[b(1-\beta)]^{\rho/\delta}\Gamma\left(\frac{\eta_1}{1-\beta} + 1 + \frac{\rho}{\delta}\right)}{\Gamma\left(\frac{\rho}{\delta}\right)\Gamma\left(\frac{\eta_1}{1-\beta} + 1\right)} \theta^{\rho-1}[1 - b(1-\beta)\theta^{\delta}]^{\frac{\eta_1}{1-\beta}}. \tag{6}$$

Note that for $\beta > 1$, the density in (5) stays in the generalized type-2 beta family of densities and for $\beta < 1$ the density in (6) belongs to the generalized type-1 beta family of densities. When $\beta \to 1$, either from the left or from the right, $P_2(\theta)$ and $P_1(\theta)$ will go to $P_3(\theta)$, where