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On a modified beta function and some applications

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Dedicated to Professor H. M. Srivastava on the Occasion of his Seventieth Birth Anniversary

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ABSTRACT

In this work, we study the modified beta function $B_{\frac{1}{2}}(x;a,b) = \int_{1/2}^{x}(1-v)^{a-1}v^{b-1}dv, x \in (0,1)$ that appeared in Marušić and Bajzer's solution [M. Marušić, Ž. Bajzer, Generalized two-parameter equation of growth, J. Math. Anal. Appl. 179 (1993) 446–462] of the generalized two-parameter equation of tumor growth $y' = ay^{\mu} + by^{\nu}$. In particular, we provide methods for computing this function for special values of the function's parameters. Some analytic solutions of the generalized two-parameter growth equation are obtained.

1. Introduction

The incomplete beta function is defined by the integral

$$B(x;a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt, \quad \Re(a) > 0, \quad \Re(b) > 0, \quad x \in [0,1]. \tag{1}$$

A history of the development and numerical evaluation of this function can be found in Dutka [2]. In the analysis of tumor growth through mathematical models, Marušić and Bajzer [3] (see also the work of Bajzer et al. [1]) proposed a modified version of this function that best fit the solution of the two-parameter growth nonlinear first-order differential equation

$$y' = ay^{\alpha} + by^{\beta}, \quad y(t_0) = y_0, \quad a, b, \alpha, \beta \in \mathbb{R}, \tag{2}$$

where the integral's lower bound in Eq. (1) was replaced by 1/2. It was referred to as a modified beta function and was defined by

$$B_{\frac{1}{2}}(x;a,b) = \int_{\frac{1}{2}}^{x}(1-v)^{a-1}v^{b-1}dv, \quad x \in (0,1). \tag{3}$$

Two particular cases of (3) were most commonly used in the analysis where the function (3) was first introduced. These cases were labeled as $\alpha_\delta(x)$ and $\gamma_\delta(x)$, for δ a real parameter, and were defined as

$$\alpha_\delta(x) = B_{\frac{1}{2}}(x;\delta+1,0) = \int_{\frac{1}{2}}^{x}(1-v)^{\delta}v^{-1}dv \tag{4}$$

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and

$$\gamma_\delta(x) = -B_{\frac{1}{2}}\left(\frac{1}{x}; \delta + 1, -\delta\right) = \int_2^x (v-1)^\delta v^{-1} dv. \tag{5}$$

Through the separation of variables technique, it is straightforward to show, for example, that in the case where $1 + \frac{b}{a} y_0^{\beta-\alpha} > 0$ and $a \cdot b < 0$, the solution of generalized growth Eq. (2) can be written as

$$y(t) = \left(-\frac{a}{b}\right)^{1/(\beta-\alpha)} \left[1 - \alpha_\delta^{-1}(\alpha_\delta(v_0) + k(t - t_0))\right]^{\frac{1}{(\beta-\alpha)}}, \tag{6}$$

where

$$v_0 = 1 + \frac{b}{a} y_0^{\beta-\alpha}, \quad k = b(\beta-\alpha)\left|\frac{a}{b}\right|^{\frac{(\beta-1)}{(\beta-\alpha)}}, \quad \delta = -\frac{\beta-1}{\beta-\alpha} \tag{7}$$

and $\alpha_\delta(\cdot)$ is given by (4). Furthermore, for $v_0 > 0$, $a \cdot b > 0$ and $y_0 > 0$, the solution of (2) can be written as,

$$y(t) = \left(\frac{a}{b}\right)^{1/(\beta-\alpha)} \left[\gamma_\delta^{-1}(\gamma_\delta(v_0) + k(t - t_0)) - 1\right]^{\frac{1}{(\beta-\alpha)}}, \tag{8}$$

where $\gamma_\delta(\cdot)$ is given in (5). The importance of the generalized growth Eq. (2) comes from the fact that it contains some of the most classical models of population growth such as the logistic ($\mu = 1, v = 2$), von Bertalanffy ($\mu = 2/3, v = 1$) and von Bertalanffy–Richards ($\mu = 1, v = n > 1$) equations. For the lack of explicit analytic expressions in terms of elementary functions and the complexity of computing the exact solutions of (2) using Eq. (3), most researchers found it more convenient to solve the nonlinear differential Eq. (2) numerically rather than evaluate complex expressions involving functions not included in standard mathematical libraries.

The purpose of the present work is to provide explicit algebraic expressions for the modified beta function (3), and its variations (4) and (5), in terms of elementary functions. These could facilitate the computations of exact solutions of the growth Eq. (2) and could serve as benchmarks to test the precisions of any numerical approximation used in solving the non-linear differential equation of type (2). It should be clear that there is an obvious relation between the modified beta function and the well-known incomplete beta function (1) given by

$$B_{\frac{1}{2}}(x; a, b) = B(x; b, a) - B\left(\frac{1}{2}; b, a\right) \tag{9}$$

or equivalently,

$$B_{\frac{1}{2}}(x; a, b) = B\left(\frac{1}{2}; a, b\right) - B(1 - x; a, b). \tag{10}$$

Thus the solution of the growth equation can be expressed, as well, in terms of $B(x; b, a)$. However, it would be tedious in construction as well as in computation. Generally speaking, the reduction formulas obtained here can also be applied to the incomplete beta function (1).

This paper is organized as follows: In Section 2, some elementary properties of the modified beta function are discussed. In Section 3, some new reduction and transformation formulas for the modified beta function are obtained. In Section 4, we discuss some summation formulas for the modified beta functions. In Section 5, we used our results to report analytic solutions of the growth Eq. (2) for different values of the parameters α and β.

2. Elementary properties of the modified beta function

This section is devoted to study some elementary properties of the modified beta function such as monotonicity and concavity. These properties are straightforward and are stated here without proofs.

Theorem 1 ([3], Lemma 1). *The modified beta function (3) is defined and strictly increasing over the interval $x \in (0,1)$ with range*

$$B_{\frac{1}{2}}(\cdot; a, b) : (0,1) \to \begin{cases} (-\infty, \infty) & : a \leqslant 0, \quad b \leqslant 0, \\ (L_{a,b}, \infty) & : a \leqslant 0, \quad b > 0, \\ (-\infty, U_{a,b}) & : a > 0, \quad b \leqslant 0, \\ (L_{a,b}, U_{a,b}) & : a > 0, \quad b > 0, \end{cases} \tag{11}$$

where $L_{a,b} = \lim_{x\to 0} B_{\frac{1}{2}}(x; a, b)$ for $b > 0$ and $U_{a,b} = \lim_{x\to 1} B_{\frac{1}{2}}(x; a, b)$ for $a > 0$.

Clearly, from this theorem, we note that

$$\alpha_\delta(\cdot) : (0,1) \to \begin{cases} (-\infty, \infty) & : \delta \leqslant -1, \\ (-\infty, U_{a,b}) & : \delta > -1 \end{cases} \tag{12}$$