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The equivalence of curves in $SL(n,R)$ and its application to ruled surfaces

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Dedicated to Professor H. M. Srivastava on the Occasion of his Seventieth Birth Anniversary

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ABSTRACT

The generating system of the differential algebra for invariant differential polynomials with two parametric curves is obtained. Conditions for the equivalence of two parametric curves families are given. We are also proved that the generating differential invariants of two parametric curves are independent. Finally, we reduce the $SL(n,R)$-equivalent problem for ruled surfaces to that of parametric curves.

1. Introduction

Integral invariants depend on quantities obtained by integration of various functions along a curve. The type of integral invariants that we consider was introduced by [4]. In this paper, we obtain, for the first time, explicit formulae of $SL(n,R)$-invariants for a pair of parametric curves. Explicit expression for integral invariants, however, appear to be known only for a single curve, as computations become challenging for a pair of parametric curves.

In the literature, there are many papers on the invariant theory of curves in differential geometry [1,2,6]. In most of the studies, special invariants were considered such as arc length, curvature and torsion. The problem of equivalence has been already solved for a single curve by Sağiroğlu [4]. In this study, we have solved the equivalence of parametric curves problem choosing a pair of parametric curves in $SL(n,R)$.

$SL(n,R)$-equivalence of a pair of parametric curves is defined similar to a single parametric curve. The standard action of the $SL(n,R)$-group on R^n induces on action on parametric curves. Then we have obtained the generator system of $SL(n,R)$ differential invariants for a pair of parametric curves and used these differential invariants in order to show the $SL(n,R)$-equivalence of two different pairs of parametric curves. So we have seen that generator invariants provide some equalities for the equivalence. Also, it is observed that generator invariants obtained are functionally independent.

Ruled surfaces are formed by a one-parameter set of lines and have been investigated extensively in classical geometry. A ruled surface is a surface swept out by a straight line moving along a curve $a(u)$. Such a surface always has a parametrization $X(u,v) = a(u) + vb(u)$ or $X(u,v) = a(v) + ub(v)$, where we call $a(u)$ the base curve, $b(u)$ the director curve. Because of their simple generation, these surfaces arise in a variety of applications [3,5].

In this paper, we have defined $SL(n,R)$-equivalence of ruled surfaces. Then, using the known parametrizations of ruled surfaces, the $SL(n,R)$-equivalence of ruled surfaces has been reduced to that of two pairs of parametric curves. Furthermore, for $SL(n,R)$-equivalence of two ruled surfaces, generator differential invariants of a pair of parametric curves are considered.

Let R be the field of real numbers and $I = (c,d)$ be an open interval of R.

Definition 1. A C^∞-function $x: I \to R^n$ will be called a parametric curve in R^n.

We denote the group of matrix whose determinant 1 with $SL(n,R)$. $SL(n,R)$ acts by $(g,x) \to gx$ on R^n, where gx is the multiplication of a matrix g and a column vector $x \in R^n$. If $x(t)$ is a parametric curve in R^n then $gx(t)$ is also a parametric curve in R^n for any $g \in SL(n,R)$.

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Definition 2. Let $\{x_1(t), x_2(t)\}$ and $\{y_1(t), y_2(t)\}$ be two parametric curve families in R^n. These parametric curve families will be called $SL(n,R)$-equivalent if there exists $g \in SL(n,R)$ such that $y_2(t) = gx_2(t)$ and $y_1(t) = gx_1(t)$ for all $t \in I$ and written $\{y_1(t), y_2(t)\} \overset{SL(n,R)}{\approx} \{x_1(t), x_2(t)\}$.

Let x be a parametric curve in R^n and $x'(t)$ be the derivative of $x(t)$. Put $x^{(0)} = x$, $x^{(n)} = (x^{(n-1)})'$. The determinant of vectors $x(t), x'(t), \ldots, x^{(n-1)}(t)$ will be denoted by $[x(t)x'(t)\cdots x^{(n-1)}(t)]$.

Definition 3. A parametric curve $x(t)$ in R^n will be called $SL(n,R)$-regular (shortly, regular) if $[x(t)x'(t)\cdots x^{(n-1)}(t)] \neq 0$ for all $t \in I$.

Definition 4. A polynomial $p(x, x', \ldots, x^{(k)}, y, y', \ldots, y^{(m)})$ of x, y and a finite number of derivatives $x, x', \ldots, x^{(k)}, y, y', \ldots, y^{(m)}$ of x, y with the coefficients from R will be called a differential polynomial of x and y. It will be denoted by $p\{x,y\}$. We denote the set of all differential polynomials of x and y by $R\{x,y\}$. It is a differential R-algebra.

Let G be a subgroup of $SL(n,R)$. A differential polynomial $p\{x,y\}$ will be called G-invariant if $p\{gx,gy\} = p\{x,y\}$ for all $g \in G$. The set of all G-invariant differential polynomials of x, y will be denoted by $R\{x,y\}^G$. It is a differential subalgebra of $R\{x,y\}$. A subset S of $R\{x,y\}^G$ will be called a generating system of $R\{x,y\}^G$ if the smallest differential subalgebra containing S is $R\{x,y\}^G$.

2. The generating system of $R\{x_1,x_2\}^G$

Lemma 1. *For any vectors $x_0, x_1, \ldots, x_n, y_2, \ldots, y_n$ in R^n, the following equality holds:*

$$F(x_0, x_1, \ldots, x_n) = [x_1x_2\cdots x_n][x_0y_2\cdots y_n] - [x_0x_2\cdots x_n][x_1y_2\cdots y_n] - \cdots - [x_1x_2\cdots x_0][x_ny_2\cdots y_n] = 0. \tag{1}$$

Proof. Page 46 in [1]. □

Theorem 1. *Let x_1 and x_2 be two parametric curves such that x_1 is regular. Then the generator set of $R\{x_1,x_2\}^G$ is*

$$\begin{aligned}&\left[x_1x_1'\cdots x_1^{(n-1)}\right], \left[x_1\cdots x_1^{(i-1)}x_1^{(n)}x_1^{(i+1)}\cdots x_1^{(n-1)}\right], \quad i = 0, \ldots, n-2,\\ &\left[x_1\cdots x_1^{(i-1)}x_2x_1^{(i+1)}\cdots x_1^{(n-1)}\right], \quad i = 0, \ldots, n-1.\end{aligned} \tag{2}$$

Proof. By the first main theorem for $SL(n,R)$ in [7], the generator set of $R(x_\tau, \tau \in \Delta)^G(\text{for}|\Delta| \geqslant n)$ is

$$[x_1x_2\cdots x_n], \quad \left[x_1\cdots x_{i-1}x_\tau x_{i+1}\cdots x_1^{(n-1)}\right], \quad i = 1, \ldots, n, \quad \tau \in \Delta/\{1, \ldots, n\}.$$

We take $x_1, x_2, x_1', x_2', \ldots, x_1^{(K)}, x_2^{(K)}, \ldots$ instead of x_τ, then the generator set of $R(x_1, x_2, x_1', x_2', \ldots, x_1^{(K)}, x_2^{(K)}, \ldots)^G$ is

$$\Big\{\left[x_1 \quad x_1' \quad \cdots \quad x_1^{(n-1)}\right], \quad \left[x_1 \quad \cdots \quad x_1^{(i-1)} \quad x_1^{(\tau)} \quad x_1^{(i+1)} \quad \cdots \quad x_1^{(n-1)}\right], \quad \tau \geqslant n$$

$$\left[x_1 \quad \cdots \quad x_1^{(i-1)} \quad x_2^{(\tau)} \quad x_1^{(i+1)} \quad \cdots \quad x_1^{(n-1)}\right], \tau \geqslant 0\Big\}.$$

We obviously have

$$\left[x_1 \quad x_1' \quad \cdots \quad x_1^{(n-1)}\right]' = \left[x_1 \quad \cdots \quad x_1^{(n-2)} \quad x_1^{(n)}\right]. \tag{3}$$

Firstly, we want to show that

$$\left[x_1 \quad \cdots \quad x_1^{(i-1)} \quad x_1^{(\tau)} \quad x_1^{(i+1)} \quad \cdots \quad x_1^{(n-1)}\right], \quad (\tau \geqslant n)$$

are generated by

$$\left[x_1 \quad \cdots \quad x_1^{(i-1)} \quad x_1^{(n)} \quad x_1^{(i+1)} \quad \cdots \quad x_1^{(n-1)}\right], \quad (i = 0, \ldots, n-2).$$

Since (3), for $\tau = n$

$$\left[x_1 \quad \cdots \quad x_1^{(i-1)} \quad x_1^{(n)} \quad x_1^{(i+1)} \quad \cdots \quad x_1^{(n-1)}\right], \quad i = 0, \ldots, n-1$$

is generated by set (2). Let $\tau > n$. By the induction hypothesis, for $\tau - 1$

$$\left[x_1 \quad \cdots \quad x_1^{(i-1)} \quad x_1^{(\tau-1)} \quad x_1^{(i+1)} \quad \cdots \quad x_1^{(n-1)}\right]$$