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Singular perturbation degenerate problems occurring in atmospheric dispersion of pollutants

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Dedicated to Professor H. M. Srivastava on the Occasion of his Seventieth Birth Anniversary

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ABSTRACT

The boundary value problems for the degenerate differential-operator equations with small parameters generated on all boundary are studied. Several conditions for the separability and the fredholmness in Banach-valued L_p-spaces of are given. In applications, maximal regularity of degenerate Cauchy problem for parabolic equation arising in atmospheric dispersion of pollutants studied.

1. Introduction, notations and background

The main objective of the present paper is to discuss singular perturbation boundary value problems (BVPs) for degenerate differential operator equation (DOE)

$$Lu = -\varepsilon u^{[2]}(x) + Au(x) + \varepsilon^{\frac{1}{2}} A_1(x) u^{[1]}(x) + A_2(x)u(x) = f,$$

on $(0,1)$, where

$$D_x^{[i]} u = u^{[i]}(x) = \left[x^{\gamma_1}(1-x)^{\gamma_2} \frac{d}{dx} \right]^i u(x)$$

and ε is a small parameter.

In applications maximal regularity properties of Cauchy problem for the following degenerate parabolic equation with small parameter

$$u_t - \varepsilon D_x^{[2]} u(t,x) + Au(t,x) + \varepsilon^{\frac{1}{2}} A_1(x) u^{[1]}(t,x) + A_2(x) u(t,x) = f(y,x), \quad t \in R_+, \; x \in (0,1)$$

is established, where A, A_1, A_2 are linear operators in a Banach space E. The above singular perturbation problems occur in different situation of fluid mechanics environmental engineering et.s.

Note that, DOEs are studied e.g. in [1–9] and [11,12].

Let $\gamma = \gamma(x)$, $x = (x_1, x_2, \ldots, x_n)$ be a positive measurable function on a domain $\Omega \subset R^n$. Let $L_{p,\gamma}(\Omega; E)$ denote the space of strongly measurable E-valued functions that are defined on Ω with the norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\Omega;E)} = \left(\int \|f(x)\|_E^p \gamma(x) dx \right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty.$$

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For $\gamma(x)\equiv 1$ the space $L_{p,\gamma}(\Omega;E)$ will be denoted by $L_p = L_p(\Omega;E)$.

Let $E(A^\theta)$ denote the space $D(A^\theta)$ with norm

$$\|u\|_{E(A^\theta)} = (\|u\|^p + \|A^\theta u\|^p)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty,\ 0 < \theta < \infty.$$

Consider the BVP for DOE

$$\begin{aligned}
&(L+\lambda)u = -\varepsilon u^{(2)}(x) + (A+\lambda)u(x) = f, x\in(0,b)\\
&L_1 u = \sum_{k=0}^{m_1}\varepsilon^{\sigma_k}\left[\alpha_k u^{(k)}(0) + \sum_{i=1}^{N}\delta_{ki}u^{(k)}(x_{ki})\right] = f_1\\
&L_2 u = \sum_{k=0}^{m_2}\varepsilon^{\sigma_k}\left[\beta_k u^{(k)}(b) + \sum_{i=1}^{N}\nu_{ki}u^{(k)}(x_{ki})\right] = f_2,
\end{aligned} \tag{1}$$

where $m_k\in\{0,1\};\ \sigma_k = \frac{k}{2}+\frac{1}{2p},\ \alpha_k,\ \beta_k,\ \delta_{ki},\ \nu_{ki}$ are complex numbers and $x_{ki}\in(0,b)$; A is a possible unbounded operator in E and $A_\lambda = A+\lambda;\ 0\leqslant\gamma<1,\ f_j\in X_j = (E(A),E)_{\theta_j,p},\ \theta_j = \frac{m_j+\frac{1-\gamma}{p}}{2}$, here $(E(A),E)_{\theta_j,p}$ are interpolation spaces obtained from $\{E(A),E\}$ by the K-method [10, Section 1.3.2]. For definitions see [8,9], for instance.

From [8] we obtain

Theorem A_1. *Let the following conditions be satisfied:*

(1) $\gamma = x^{\gamma_1}(b-x)^{\gamma_2},\ 0\leqslant\gamma_1,\ \gamma_2<1-\frac{1}{p},\ p\in(1,\infty),\ j=1,2,\dots,m-1,\ 0\leqslant\mu\leqslant 1-\frac{j}{m}$;
(2) *E is a Banach space satisfying the multiplier condition with respect to p and weighted function γ, $p\in(1,\infty)$, $0<\varepsilon\leqslant T<\infty$ and $0<h\leqslant h_0<\infty$ are certain parameters;*
(3) *A is an R-positive operator in E;*

Then, the embedding

$$D^j W^m_{p,\gamma}(0,b;E(A),E)\subset L_{p,\gamma}\left(0,b;E\left(A^{1-\frac{j}{m}-\mu}\right)\right)$$

is continuous and there exists a positive constant C_μ such that

$$h^{\frac{j}{m}}\|u^{(j)}\|_{L_{p,\gamma}(0,b;E(A^{1-\frac{j}{m}-\mu}))} \leqslant C_\mu\left[h^\mu\|u\|_{W^m_{p,\gamma}(0,b;E(A),E)} + h^{-(1-\mu)}\|u\|_{L_{p,\gamma}(0,b;E)}\right]$$

for all $u\in W^m_{p,\gamma}(0,b;E(A),E)$ and h;

If $A^{-1}\in\sigma_\infty(E)$ and $0<\mu\leqslant 1-\frac{j}{m}$ then the embedding

$$D^j W^m_{p,\gamma}(a,b;E(A),E)\subset L_{p,\gamma}\left(a,b;E\left(A^{1-\frac{j}{m}-\mu}\right)\right)$$

is compact.

In a similar way as [9] we obtain.

Theorem A_2. *Let the following conditions be satisfied:*

(1) *α_k, β_k, δ_{kj} are complex numbers, $\alpha_{m_k}\neq 0, \beta_{m_k}\neq 0,\ a>0,\ t$ is a small positive parameter;*
(2) *E is the Banach space satisfying the multiplier condition with respect to p and weighted function*

$$\gamma(x) = x^\gamma, \quad 0\leqslant\gamma<1-\frac{1}{p}, \quad 1<p<\infty;$$

(3) *A is an R positive operator in E.*

Then, the problem (1) for all $f\in L_{p,\gamma}(0,b;E)$ and $f_j\in$ has a unique solution $u\in W^2_{p,\gamma}(0,b;E(A),E)$. Moreover for $|\arg\lambda|\leqslant\varphi$ and sufficiently large $|\lambda|$ the following uniform coercive estimate holds

$$\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\varepsilon^{\frac{i}{2}}\|u^{(i)}\|_{L_{p,\gamma}(0,b;E)} + \|Au\|_{L_{p,\gamma}(0,b;E)} \leqslant C\left[\|f\|_{L_p(0,b;E)} + \sum_{j=1}^{2}\|f_j\|_{X_j}\right].$$

Consider the BVP for the degenerate differential-operator equation with small parameter

$$\begin{aligned}
&Lu = -\varepsilon u^{[2]}(x) + Au(x) + \varepsilon^{\frac{1}{2}}A_1(x)u^{[1]}(x) + A_2(x)u(x) = f, \quad x\in(0,1),\\
&L_1 u = \sum_{i=0}^{m_1}\varepsilon^{\sigma_i}\alpha_i u^{[i]}(0) = 0, \quad L_2 u = \sum_{i=0}^{m_2}\varepsilon^{\sigma_i}\beta_i u^{[i]}(1) = 0,
\end{aligned} \tag{2}$$