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Three solutions for a p-Laplacian boundary value problem with impulsive effects ☆

Liang Bai, Binxiang Dai *

School of Mathematical Sciences and Computing Technology, Central South University, Changsha, Hunan 410075, People's Republic of China

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ABSTRACT

In this paper, a p-Laplacian boundary value problem with impulsive effects is considered. Multiplicity of solutions is obtained by three critical points theorem. An example is presented to illustrate main result.

1. Introduction

In this paper, we are interested in ensuring the existence of at least three solutions for the following p-Laplacian boundary value problem

$$(\rho(t)\Phi_p(u'(t)))' - s(t)\Phi_p(u(t)) + \lambda f(t,u(t)) = 0, \quad \text{a.e. } t \in (a,b), \tag{1}$$

$$\alpha_1 u'(a^+) - \alpha_2 u(a) = 0, \quad \beta_1 u'(b^-) + \beta_2 u(b) = 0, \tag{2}$$

with the impulsive conditions

$$\Delta(\rho(t_j)\Phi_p(u'(t_j))) = I_j(u(t_j)), \quad j = 1,2,\ldots,l, \tag{3}$$

where $\Phi_p(x):=|x|^{p-2}x$, $p > 1$, $a < b$, $\rho, s \in L^\infty([a,b])$ with $\text{ess inf}_{[a,b]}\rho > 0$ and $\text{ess inf}_{[a,b]}s > 0$, $\rho(a^+) = \rho(a) > 0$, $\rho(b^-) = \rho(b) > 0$, α_1, α_2, β_1, β_2 are positive constants, $f:[a,b] \times \mathbf{R} \to \mathbf{R}$ is continuous, $t_0 = a < t_1 < t_2 < \cdots < t_l < t_{l+1} = b$, I_j: $\mathbf{R} \to \mathbf{R}$, $j = 1, 2, \ldots, l$ are continuous, $\lambda \in [0,+\infty)$ is a parameter and

$$\Delta(\rho(t_j)\Phi_p(u'(t_j))) = \rho(t_j^+)\Phi_p(u'(t_j^+)) - \rho(t_j^-)\Phi_p(u'(t_j^-)),$$

where $z(y^+)$ and $z(y^-)$ denote the right and left limits of $z(y)$ at y, respectively. We refer to impulsive problem (1)–(3) as (IP).

Differential equations with impulsive effects arising from the real world describe the dynamics of processes in which sudden, discontinuous jumps occur. For the background, theory and applications of impulsive differential equations, we refer the interest readers to [1–7]. There have been many approaches to study the existence of solutions of impulsive differential equations, such as fixed point theory, topological degree theory (including continuation method and coincidence degree theory) and comparison method (including upper and lower solutions methods and monotone iterative method) and so on (see, for example, [8–11] and references therein).

Recently, [12–22] using variational method studied the existence and multiplicity of solutions of impulsive problem. More precisely, in [13] Tian and Ge obtained sufficient conditions that guarantee the existence of at least two positive solu-

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* Corresponding author.
E-mail address: bxdai@mail.csu.edu.cn (B. Dai).

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tions of a p-Laplacian boundary value problem with impulsive effects. Two key conditions of the main result of [13] are listed as follows.

(C1) There exist $\mu > p$, $h \in C([a,b] \times [0,+\infty), [0,+\infty))$, $\eta > 0$, $r \in C([a,b], [0,+\infty))$, $g \in C([0,+\infty), [0,+\infty))$ and

$$\int_a^b r(s)ds + \eta > 0,$$

such that

$$f(t,x) = r(t)\Phi_\mu(x) + h(t,x), \quad I_i(x) = \eta\Phi_\mu(x) + g(x).$$

(C2) There exist $c \in L^1([a,b], [0,+\infty))$, $d \in C([a,b], [0,+\infty))$, $\xi \geqslant 0$, such that

$$h(t,x) \leqslant c(t) + d(t)\Phi_p(x), \quad g(x) \leqslant \xi\Phi_p(x).$$

However, there are many cases which can't be dealt with by the result of [13]. For example, $p = 3$, there is only one impulsive point $t_1 \in (a,b)$ and the impulsive condition is $-\Delta(\rho(t_1)\Phi_3(u'(t_1))) = G(u(t_1))$ where

$$G(x) = \frac{1}{12} + \frac{5}{24}|x|^{\frac{3}{2}}. \tag{4}$$

In fact, $\lim_{x\to 0^+} G(x)/\Phi_3(x) = +\infty$. However $(C1)$ and $(C2)$ imply that

$$\lim_{x\to 0^+} \frac{I_i(x)}{\Phi_p(x)} \leqslant \lim_{x\to 0^+} \eta|x|^{\mu-p} + \xi = \xi.$$

In [23], Ricceri established a three critical points theorem. After that, the theorem has been used extensively and a series of results are obtained (see, for example, [24–28]).

Existence and multiplicity of solutions for p-Laplacian boundary value problem have been studied extensively in the literature (see, for example, [24,29–32] and references therein). However, to the best of our knowledge, existence of at least three solutions for p-Laplacian boundary value problem with impulsive effects has attracted less attention.

Motivated by the above facts, in this paper we devote to study the multiplicity of solutions of (IP) via three critical points theorem obtained by Ricceri [23].

Throughout this paper, we assume that.

(H1) There exist constants $a_j > 0$, $b_j > 0$ and $\gamma_j \in [0, p-1)$, $j = 1, 2, \ldots, l$ such that

$$|I_j(x)| \leqslant a_j + b_j|x|^{\gamma_j} \quad \text{for every } x \in \mathbf{R}, \quad j = 1, 2, \ldots, l.$$

The remaining part of this paper is organized as follows. Some fundamental facts will be given in Section 2. In Section 3, main result of this paper will be presented and an example will be given to illustrate the theorem.

In the following, for convenience, when we make a statement without specifying the domain of λ, we assume that it holds for all $\lambda \in [0,+\infty)$.

2. Preliminaries

Here and in the sequel, U will denote the Sobolev space $W^{1,p}([a,b])$ equipped with the norm

$$\|u\| = \left(\int_a^b \rho(t)|u'(t)|^p + s(t)|u(t)|^p dt\right)^{1/p},$$

which is equivalent to the usual one. We define the norm in $C([a,b])$ as $\|u\|_\infty = \max_{t\in[a,b]}|u(t)|$.

Consider J: $U \times [0,+\infty) \to \mathbf{R}$ defined by

$$J(u,\lambda) = \varphi_1(u) + \lambda\varphi_2(u),$$

where

$$\varphi_1(u) = \frac{\|u\|^p}{p} + \sum_{j=1}^{l} \int_0^{u(t_j)} I_j(x)dx + \frac{\rho(a)\alpha_2^{p-1}}{p\alpha_1^{p-1}}|u(a)|^p + \frac{\rho(b)\beta_2^{p-1}}{p\beta_1^{p-1}}|u(b)|^p,$$

$$\varphi_2(u) = -\int_a^b F(t,u(t))dt$$

and

$$F(t,x) = \int_0^x f(t,y)dy \quad \text{for all } (t,x) \in [a,b] \times \mathbf{R}.$$