Applied Mathematics and Computation 217 (2011) 10024–10034

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Dynamics of a competitive Lotka–Volterra system with three delays ☆

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ARTICLE INFO

Keywords:
Lotka–Volterra system
Hopf bifurcation
Delay
Periodic solution

ABSTRACT

In this paper, a competitive Lotka–Volterra system with three delays is investigated. By choosing the sum τ of three delays as a bifurcation parameter, we show that in the above system, Hopf bifurcation at the positive equilibrium can occur as τ crosses some critical values. And we obtain the formulae determining direction of Hopf bifurcation and stability of the bifurcating periodic solutions by using the normal form theory and center manifold theorem. Finally, numerical simulations supporting our theoretical results are also included.

1. Introduction

The n-species Lotka–Volterra competition system with delays can be modeled by the following system

$$\dot{x}_i(t) = x_i(t)\left[r_i - \sum_{j=1}^{n} a_{ij}x_j(t-\tau_{ij})\right], \quad i = 1,2,\ldots,n,$$

where r_i, a_{ij}, τ_{ij} $(i,j = 1,2,\ldots,n)$ are positive constants, and x_i $(i = 1,2,\ldots,n)$ can be interpreted as the densities of certain species. In the absence of interspecific interactions, the species is governed by the well known logistic equation $\dot{x}(t) = x(t)[r - kx(t)]$. In the presence of interactions, each species restrains the average growth rate of the other and has the corresponding delay.

Recently, there have been extensive literatures dealing with the above system or systems similar to the above system, regarding attractivity, persistence, global stabilities of equilibrium and other dynamics (see, for example, [1,2,14–19] and references therein). For a long time, it has been recognized that delays can have very complicated impact on the dynamics of a system (see, for example, monographes by Hale and Lunel [4], Kuang [6] and Wu [9]). For example, delays can cause the loss of stability and can induce various oscillations and periodic solutions through the Hopf bifurcation in delay differential equations, and the study on the stability and local Hopf bifurcation of systems similar to the above system can be seen in [3–13].

In this paper, we consider the following three-species Lotka–Volterra competition system with discrete delays described by

☆ Supported partly by NNSF of China (Grant Nos. 10771215, 10771094), Project of Hunan Provincial Education Department (Grant No. 07C639) and Project of Hunan Provincial Youth Key Teacher.

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doi:10.1016/j.amc.2011.04.078

$$\begin{cases} \dot{x}_1(t) = x_1(t)[r_1 - a_{11}x_1(t) - a_{13}x_3(t-\tau_3)], \\ \dot{x}_2(t) = x_2(t)[r_2 - a_{21}x_1(t-\tau_1) - a_{22}x_2(t)], \\ \dot{x}_3(t) = x_3(t)[r_3 - a_{32}x_2(t-\tau_2)] \end{cases} \tag{1.1}$$

and the initial conditions

$$\dot{x}_i(t) = \Phi_i(t) \geqslant 0, \quad t \in [-\tau, 0), \quad \Phi_i(0) > 0, \quad \tau = \tau_1 + \tau_2 + \tau_3, \quad i = 1, 2, 3, \tag{1.2}$$

where $x_1(t), x_2(t), x_3(t)$ denote the density of species at time t, respectively; τ_i $(i = 1, 2, 3) \geqslant 0$ is the feedback time delay of species $x_i(t)$ $(i = 1, 2, 3)$ to the growth of species itself; r_i $(i = 1, 2, 3) > 0$ is the intrinsic growth rate of the ith species and $\frac{a_{ij}}{r_i} > 0 (i, j = 1, 2, 3)$ are interaction coefficients measuring the extent to which the jth species affects the growth rate of the ith species.

Considered the biological interpretation of system (1.1), there is always a unique positive equilibrium $E_* = (x_1^*, x_2^*, x_3^*)$ provided that the condition

$$(H_1) a_{32}r_2 > a_{22}r_3,$$
$$(H_2) a_{21}a_{32}r_1 + a_{11}a_{22}r_3 > a_{11}a_{32}r_2$$

hold, where

$$x_1^* = \frac{a_{32}r_2 - a_{22}r_3}{a_{21}a_{32}},$$
$$x_2^* = \frac{r_3}{a_{32}},$$
$$x_3^* = \frac{a_{21}a_{32}r_1 + a_{11}a_{22}r_3 - a_{11}a_{32}r_2}{a_{32}a_{21}a_{13}}.$$

When the delay $\tau_1 = \tau_2 = \tau_3 = 0$, the system (1.1) simplifies to an autonomous system of ordinary differential equation of the form

$$\begin{cases} \dot{x}_1(t) = x_1(t)[r_1 - a_{11}x_1(t) - a_{13}x_3(t)], \\ \dot{x}_2(t) = x_2(t)[r_2 - a_{21}x_1(t) - a_{22}x_2(t)], \\ \dot{x}_3(t) = x_3(t)[r_3 - a_{32}x_2(t)]. \end{cases} \tag{1.3}$$

The main purpose of this paper is to investigate the effects of the delay on the solutions of system (1.1), and we mainly study the stability, the local Hopf bifurcation for system (1.1). We would like to mention that bifurcations in a population dynamics with a single delay or two delays had been investigated by many researchers (see, for example, [3–13]). However, there are few papers on the bifurcation of a population dynamics with three delays or multiple delays. Hence, the research of Hopf bifurcation for competitive Lotka–Volterra systems with three or multiple delays is worth further consideration.

The remainder of the paper is organized as follows. In Section 2, the stability of the equilibrium and the existence of Hopf bifurcation at the positive equilibrium are studied. In Section 3, the direction of Hopf bifurcation, stability and period of bifurcating periodic solutions on the center manifold are determined. Numerical simulations supporting our theoretical results are also included in Section 4. Finally, we give some biological explanations and conclusions.

2. Stability of the positive equilibrium and existence of local Hopf bifurcation

In this section, we always have the following assumption.

$$(H_3)(a_{11}^2a_{22}a_{32} + a_{11}a_{21}a_{32}^2)r_2 + a_{11}a_{21}a_{22}^2r_3 > a_{21}^2a_{32}^2r_1 + (a_{11}^2a_{22}^2 + a_{11}a_{22}a_{21}a_{32})r_3.$$

For convenience, let us introduce new variables $u_1(t) = x_1(t - \tau_1 - \tau_2), u_2(t) = x_2(t - \tau_2), u_3(t) = x_3(t), \tau = \tau_1 + \tau_2 + \tau_3$ so that system (1.1) can be written as the following equivalent system with a single delay:

$$\begin{cases} \dot{u}_1(t) = u_1(t)[r_1 - a_{11}u_1(t) - a_{13}u_3(t-\tau)], \\ \dot{u}_2(t) = u_2(t)[r_2 - a_{21}u_1(t) - a_{22}u_2(t)], \\ \dot{u}_3(t) = u_3(t)[r_3 - a_{32}u_2(t)]. \end{cases} \tag{2.1}$$

Under the hypothesis (H_1, H_2, H_3), let $v_1(t) = u_1(t) - x_1^*$, $v_2(t) = u_2(t) - x_2^*$, $v_3(t) = u_3(t) - x_3^*$, the system (2.1) can be rewritten as the following equivalent system:

$$\begin{cases} \dot{v}_1(t) = (v_1(t) + x_1^*)[-a_{11}v_1(t) - a_{13}v_3(t-\tau)], \\ \dot{v}_2(t) = (v_2(t) + x_2^*)[-a_{21}v_1(t) - a_{22}v_2(t)], \\ \dot{v}_3(t) = (v_3(t) + x_3^*)[-a_{32}v_2(t)]. \end{cases}$$

Hence, the positive equilibrium $E_*(x_1^*, x_2^*, x_3^*)$ of system (1.1) is transformed into zero equilibrium of the above system. Linearizing the above system about the equilibrium $(0, 0, 0)$ and replacing $v_i(t)$ with $u_i(t)$ $(i = 1, 2, 3)$, we can get the following linear system.