Applied Mathematics and Computation 217 (2011) 9543–9551

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A double projection algorithm for multi-valued variational inequalities and a unified framework of the method ☆

Changjie Fang [a,*], Yiran He [b]

[a] Institute of Applied Mathematics, Chongqing University of Posts and Telecommunications, Chongqing 400065, China
[b] Department of Mathematics, Sichuan Normal University, Chengdu, Sichuan 610068, China

ARTICLE INFO

Keywords:
Generalized variational inequality
Pseudomonotone mapping
Multi-valued mapping
Projection algorithm
Unified framework

ABSTRACT

In this paper, we propose a double projection algorithm for a generalized variational inequality with a multi-valued mapping. Under standard conditions, our method is proved to be globally convergent to a solution of the variational inequality problem. Moreover, we present a unified framework of projection-type methods for multi-valued variational inequalities. Preliminary computational experience is also reported.

1. Introduction

We consider the following generalized variational inequality: to find $x^* \in C$ and $\xi \in F(x^*)$ such that

$$\langle \xi, y - x^* \rangle \geqslant 0, \quad \forall y \in C, \tag{1}$$

where C is a nonempty closed convex set in $\mathbb{R}^n$, F is a multi-valued mapping from C into $\mathbb{R}^n$ with nonempty values, and $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the inner product and norm in $\mathbb{R}^n$, respectively.

Theory and algorithm of generalized variational inequality have been much studied in the literature [1–9]. Various algorithms for computing the solution of (1) are proposed. The well-known proximal point algorithm [10] requires the multi-valued mapping F be monotone. Relaxing the monotonicity assumption, [1] proved if the set C is a box and F is order monotone, then the proximal point algorithm still applies for the problem (1). Assume that F is pseudomonotone, [11] described a combined relaxation method for solving (1); see also [12,13]. Projection-type algorithms have been extensively studied in the literature, see [14–16] and the references therein. Here we will devise a double projection algorithm for generalized variational inequality and prove the global convergence of the generalized iteration sequence, assuming that F is pseudomonotone in the sense of Karamardian [17]. At the same time, we present a unified framework of projection-type method for multi-valued variational inequalities and show that the framework is globally convergent under mild assumption. Furthermore, if F is a single-valued mapping, this framework contains as special cases the double projection methods for the corresponding single-valued variational inequalities.

Let S be the solution set of (1), that is, those points $x^* \in C$ satisfying (1). Throughout this paper, we assume that the solution set S of the problem (1) is nonempty and F is continuous on C with nonempty compact convex values satisfies the following property:

$$\langle \zeta, y - x \rangle \geqslant 0, \quad \forall y \in C, \quad \zeta \in F(y), \quad \forall x \in S. \tag{2}$$

☆ This work is partially supported by Natural Science Foundation Project of CQ CSTC (No. 2010BB9401) and the National Natural Science Foundation of China (No. 10701059).

* Corresponding author at: Institute of Applied Mathematics, Chongqing University of Posts and Telecommunications, Chongqing 400065, China.
E-mail address: fangcj@cqupt.edu.cn (C. Fang).

0096-3003/$ - see front matter
doi:10.1016/j.amc.2011.04.009

The property (2) holds if F is pseudomonotone on C in the sense of Karamardian. In particular, if F is monotone, then (2) holds.

The organization of this paper is as follows. In the next section, we recall the definition of continuous multi-valued mapping and present the algorithm details and prove several preliminary results for convergence analysis in Section 3. We give a unified framework of projection-type algorithm for multi-valued variational inequalities in Section 4. Numerical results are reported in the last section.

2. Algorithms

Let us recall the definition of continuous multi-valued mappings. F is said to be upper semicontinuous at $x \in C$ if for every open set V containing $F(x)$, there is an open set U containing x such that $F(y) \subset V$ for all $y \in C \cap U$. F is said to be lower semicontinuous at $x \in C$ if give any sequence x_k converging to x and any $y \in F(x)$, there exists a sequence $y_k \in F(x_k)$ that converges to y. F is said to be continuous at $x \in C$ if it is both upper semicontinuous and lower semicontinuous at x. If F is single-valued,then both upper semicontinuity and lower semicontinuity reduce to the continuity of F.

Let Π_C denote the projector onto C and let $\mu > 0$ be a parameter.

Proposition 2.1. *$x \in C$ and $\xi \in F(x)$ solves the problem (1) if and only if*

$$r_\mu(x, \xi) := x - \Pi_C(x - \mu\xi) = 0.$$

Algorithm 1. Choose $x_0 \in C$ and three parameters $\sigma > 0$, $\mu \in (0, 1/\sigma)$ and $\gamma \in (0, 1)$. Set $i = 0$.

Step 1. If $r_\mu(x_i, \xi) = 0$ for some $\xi \in F(x_i)$, stop; else take arbitrarily $\xi_i \in F(x_i)$.
Step 2. Let k_i be the smallest nonnegative integer satisfying

$$\inf_{y \in F\left(x_i - \gamma^{k_i} r_\mu(x_i, \xi_i)\right)} \langle \xi_i - y, r_\mu(x_i, \xi_i)\rangle \leqslant \sigma \|r_\mu(x_i, \xi_i)\|^2. \tag{3}$$

Set $\eta_i = \gamma^{k_i}$ and $z_i = x_i - \eta_i r_\mu(x_i, \xi_i)$.
Step 3. Compute $x_{i+1} := \Pi_{C_i}(x_i)$, where $C_i := \{x \in C : h_i(x) \leqslant 0\}$ and

$$h_i(x) := \sup_{\xi \in F(z_i)} \langle \xi, x - z_i\rangle. \tag{4}$$

Let $i := i + 1$ and go to Step 1.

Remark 2.1. Let us compare the above algorithm with Algorithm 1 in [15]. First, ξ_i can be taken arbitrarily in our method. In [15], choosing ξ_i needs solving a single-valued variational inequality and hence is computationally expensive. Furthermore, our method only requires two projections at each iteration and Algorithm 1 in [15] used three ones. In addition, Armijo-type linesearch procedures in the two algorithms are also different.

We show that Algorithm 1 is well-defined and implementable.

Proposition 2.2. *If x_i is not a solution of the problem (1), then there exist a nonnegative integer k_i satisfying (3).*

Proof. Suppose that for all k and all $y \in F(x_i - \gamma^k r_\mu(x_i, \xi_i))$ we have $\langle \xi_i - y, r_\mu(x_i, \xi_i)\rangle > \sigma\|r_\mu(x_i, \xi_i)\|^2$. Since F is lower semicontinuous, $\xi_i \in F(x_i)$ and $\lim_{k\to\infty}(x_i - \gamma^k r_\mu(x_i, \xi_i)) = x_i$, there exists a sequence $y_k \in F(x_i - \gamma^k r_\mu(x_i, \xi_i))$ such that $\lim_{k\to\infty} y_k = \xi_i$. We have $\langle \xi_i - y_k,\ r_\mu(x_i, \xi_i)\rangle > \sigma\|r_\mu(x_i, \xi_i)\|^2$, for each k. Hence $\|\xi_i - y_k\| > \sigma\|r_\mu(x_i, \xi_i)\|$, for each k. Let $k \to \infty$,we have $0 = \|\xi_i - \xi_i\| \geqslant \sigma\|r_\mu(x_i, \xi_i)\| > 0$. This contradiction completes the proof. □

Lemma 2.1. *For every $x \in C$ and $\xi \in F(x)$,*

$$\langle \xi, r_\mu(x, \xi)\rangle \geqslant \mu^{-1}\|r_\mu(x, \xi)\|^2.$$

Proof. See [15, Lemma 2.3]. □

Lemma 2.2. *The function h_i defined by (4) is Lipschitz on $\mathbb{R}^n$.*

Proof. See [15, Lemma 2.2]. □