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A numerical solution to nonlinear second order three-point boundary value problems in the reproducing kernel space

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ABSTRACT

In this paper, a new numerical algorithm is provided to solve nonlinear three-point boundary value problems in a very favorable reproducing kernel space which satisfies all boundary conditions. Its reproducing kernel function is discussed in detail. We also prove that the approximate solution and its first and second order derivatives all converge uniformly. The numerical experiments show that the algorithm is quite accurate and efficient for solving nonlinear second order three-point boundary value problems.

1. Introduction

Boundary value problems for nonlinear differential equations arise in the mathematical modeling of viscoelastic and inelastic flows, deformation of beams and plate deflection theory [1–3]. Multi-point nonlinear boundary value problems, which take into account the boundary data at intermediate points of the interval under consideration, have been addressed by many authors [4–8]. Despite of the large amount of works which are done on the theoretical aspects of these kind of equations (please see [4–7,9–11] and references therein), few works are available on the numerical analysis. Authors of [12] solved numerically multi-point boundary value problems by using the Sinc-collocation method. In [13–20], finite difference methods have been proposed for the numerical solution of various nonlocal boundary value problem. Adomian decomposition method [21] and He's variational iteration method [22] are employed for solving multi-point boundary value problems.

In [7], the existence and uniqueness of solutions of the following second-order three-point boundary value problems have been studied by the monotone iterative method

$$\begin{cases} x''(t) + f(t,x) = 0, \quad t \in (0,1), \\ x'(0) = 0, \quad x(1) = \beta x(\eta), \end{cases} \tag{1.1}$$

where $f \in C(I \times \mathbb{R}, \mathbb{R}),\ I = [0,1],\ 0 < \eta < 1,\ \beta > 0$.

In [8], the authors investigated the numerical solutions of singular second order three-point boundary value problems as follows:

$$\begin{cases} a(t)x''(t) + b(t)x'(t) + c(t)x(t) = f(t,x), \quad t \in (0,1), \\ x(0) = 0, \quad x(1) = \alpha x(\eta) + \gamma, \end{cases} \tag{1.2}$$

where $a(0) = 0$ or $a(1) = 0$ and $\eta \in (0,1),\ \alpha > 0$. Besides, the authors also imposed many restrictive conditions on $a(t),\ b(t)$ and $c(t)$.

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In this work, a numerical method will be given for Eqs. (1.1) and (1.2). It is worth mentioning that we only need to assume that $a(t), b(t), c(t) \in L^2[0,1]$ for Eq. (1.2). Without loss of generality, we may put $\gamma = 0$. The proposed numerical method depends on a new reproducing kernel space. The construction of reproducing kernel space is innovative. Its reproducing kernel function has a lot of good properties that are beneficial to calculation, especially, the new reproducing kernel space satisfies all boundary conditions.

The rest of the paper is organized as follows. A new reproducing kernel space for solving problem Eq. (1.1) is constructed in Section 2; In Section 3, based on the reproducing kernel function and the minimum-point of function, the numerical algorithm is discussed. In Section 4, some numerical examples are shown to verify the effect of our method. Also a conclusion is given in Section 5.

2. A constructive method for the reproducing kernel space $H[0,1]$

In order to solve Eq. (1.1), a reproducing kernel space is defined by

$$H[0,1] = \{x(t) | x'(t) \text{ is absolutely continuous}, \ x'(0) = 0, \ x(1) = \beta x(\eta), \ x''(t) \in L^2[0,1].\}$$

The inner product and norm of $H[0,1]$ are defined by

$$\langle x(t), y(t) \rangle = x''(0)y''(0) + \int_0^1 x'''(t)y'''(t)dt, \|x(t)\| = \sqrt{\langle x, x \rangle}. \tag{2.1}$$

Lemma 2.1. *The function space $H[0,1]$ is a reproducing kernel space.*

The proof can be found in [23]. It is very important to obtain the representation of reproducing kernel function, since it is the base of our algorithm. By Lemma 2.1, there exists a reproducing kernel function in $H[0,1]$. Now, we will give the expression of reproducing kernel function $R_s(t)$ in $H[0,1]$. For any fixed $s \in [0,1]$ and any $x(t) \in H[0,1], \ R_s(t)$ must satisfy

$$\langle x(t), R_s(t) \rangle = x(s). \tag{2.2}$$

Applying (2.1), we have

$$\langle x(t), R_s(t) \rangle = x''(0)\frac{\partial^2 R_s(0)}{\partial t^2} + \int_0^1 x'''(t)\frac{\partial^3 R_s(t)}{\partial t^3}dt,$$

and

$$\int_0^1 x'''(t)\frac{\partial^3 R_s(t)}{\partial t^3}dt = \left[x''(t)\frac{\partial^3 R_s(t)}{\partial t^3} - x'(t)\frac{\partial^4 R_s(t)}{\partial t^4} + x(t)\frac{\partial^5 R_s(t)}{\partial t^5}\right]\Bigg|_0^1 - \int_0^1 x(t)\frac{\partial^6 R_s(t)}{\partial t^6}dt.$$

Therefore, we get

$$\langle x(t), R_s(t) \rangle = x''(0)\left[\frac{\partial^2 R_s(0)}{\partial t^2} - \frac{\partial^3 R_s(0)}{\partial t^3}\right] + \sum_{i=0}^{2}(-1)^i x^{(i)}(1)\frac{\partial^{5-i} R_s(1)}{\partial t^{5-i}} - x(0)\frac{\partial^5 R_s(0)}{\partial t^5} - \int_0^1 x(t)\frac{\partial^6 R_s(t)}{\partial t^6}dt.$$

Hence $R_s(t)$ is the solution of the following generalized differential equation

$$\begin{cases} -\frac{\partial^6 R_s(t)}{\partial t^6} = \delta(t-s), \\ \frac{\partial^2 R_s(0)}{\partial t^2} - \frac{\partial^3 R_s(0)}{\partial t^3} = 0, \\ \frac{\partial^{5-i} R_s(1)}{\partial t^{5-i}} = 0, \quad i = 0,1,2, \\ \frac{\partial^5 R_s(0)}{\partial t^5} = 0, \end{cases} \tag{2.3}$$

where δ denotes δ function. For $t \neq s$, it is known that $R_s(t)$ is the solution of the following linear homogeneous differential equation with six orders, i.e.,

$$\frac{\partial^6 R_s(t)}{\partial t^6} = 0, \tag{2.4}$$

with the boundary value conditions:

$$\begin{cases} \frac{\partial^2 R_s(0)}{\partial t^2} - \frac{\partial^3 R_s(0)}{\partial t^3} = 0, \\ \frac{\partial^{5-i} R_s(1)}{\partial t^{5-i}} = 0, \quad i = 0,1,2, \\ \frac{\partial^5 R_s(0)}{\partial t^5} = 0. \end{cases} \tag{2.5}$$