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Hybrids of the heat balance integral method

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ABSTRACT

A framework for refining and hybridizing the heat balance integral method is proposed. While showing the non-uniqueness of a combined integral method in the sense of Myers and Mitchell [T.G. Myers, S.L. Mitchell, Application of the combined integral method to Stefan problems, Applied Mathematical Modelling 35 (9) (2011) 4281–4294], it is evinced through the hybrids advanced and benchmarks undertaken, that for the class of finite Puiseux series commonly employed, there are no globally efficient exponents. Synthesis of local regions of high accuracy of these hybrids is realized through the introduction of an applicable splicing algorithm.

1. Introduction

This article concerns the defining and structural aspects of the heat balance integral method (HBIM) or that which is atimes known as the method of integral relations. We introduce new variants and hybrids of the HBIM hinged on a drawn interconnection with moments of the state equation considered. Alongside these, in their implementation, the class of admissible test profiles employed are deduced from appropriate interpolatory basis and or the form of the resultant osculating interpolation polynomials evoked by given boundary conditions.

The HBIM was introduced by Barenblatt [1] in his treatment of certain non-uniform homogeneous filtration problem governed by a class of nonlinear diffusion equation. The origin of the method is also ascribed to Dorodnitsyn, see [2,3] for instance. Goodman [4] likewise suggested the method in his treatment of phase change problems.

Immediate and subsequent contributions to the development and adaptation of the technique to partial differential equations driving physical systems include those of Dorodnitsyn [5] in the introduction of a general framework for the method of integral relations, Belotserkovskii and Chushkin [6,7] in the determination of critical Mach numbers of transonic and subsonic potential flows past a body, Avduevskii [8] in the study of certain turbulent boundary layer problems, Goodman [9] in his application of the technique to transient nonlinear heat transfer, and in the approximate analytic study of non-isothermal hydrodynamic processes in the lubrication layer of a sleeve bearing by Podol'skii [10], to name a few.

Without loss of generalization to higher dimensions, suppose the approximate-analytic solutions to an initial boundary value partial differential equation

$$F\left(u(x,t),\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)=0 \quad \text{on } (a,\delta(t))\times[0,T] \tag{1}$$

is sought. The domain of definition clearly depends on t. The HBIM consists of the assumption of a test function $\upsilon(x,t)$, say, of spatio-temporal variables and undetermined parameters, which satisfies the boundary and/or initial conditions of the governing partial differential equation (PDE) and allowing the resulting residual to vanish, that is

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$$\int_a^{\delta(t)} F\left(v(x,t), \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) dx = 0. \tag{2}$$

The residual, as in Eq. (2) above, is often taken relative to the spatial domain. This leads to a reduction to an ordinary differential equation (ODE) in the temporal variable of the PDE problem which is also akin to an averaging process [11]. Following these, the parameters in the trial functions are obtained and the approximate-analytic solution determined. An alternate viewpoint is that of regarding the residual as a zeroth moment of the PDE. In a general $(n+1)$-dimensional setting with moving boundaries and coordinates $x_j, j = 1, 2, \ldots, n$, a volume integral is taken over the changing spatial domain Ω_t, say, to have a similar residual equation

$$\int_{\Omega_t} F\left(v(x_j,t), \frac{\partial}{\partial x_j}, \frac{\partial}{\partial t}\right) dx_1 dx_2 \cdots dx_n = 0. \tag{3}$$

The strength of the HBIM lies in its simplicity and tractability. However, a major drawback of the technique lie in its accuracy which is strongly dependent on chosen test profiles. The main lines taken in addressing this concern are those of the advancement of criteria for selection of appropriate test functions and or the sharpening of the structure of the HBIM technique. To varying degrees of success, attempts have been made at creating formalisms for test profile selection. For instance, Volkov et al. [12] proposes a generalization of the HBIM impinging strongly on profile creation, Mosally et al. [13] motivates Gaussian profiles based on observations, Hristov [14], Myers [15,16], Mitchell and Myers [17] propound systematic approaches for realizing optimal exponents for parabolic profiles, Layeni and Adegoke [18] introduces logistic profiles motivated by consideration of phase change systems as those with competitive regimes, while Layeni and Adegoke [19] construct profiles induced by properties of heat polynomials. Excellent reviews of current trends and approaches as regards various aspects of the integral method are given in the works of Wood [9], Hristov [14], Mitchell et al. [20] and references within.

The original idea of the HBIM involves the setting of the residual accrued by pertinent test profiles, with undetermined parameters, to naught. This is equivalent to making the zeroth moment of the state expression over the moving domain vanish. With a view to improving the accuracy of the HBIM, Volkov and Li-Orlov [21] introduced the idea of taking a double integration in the treatment of the problem of transient heat conduction without phase change. This idea, which in its adaptation to Stefan problems by Sadoun [22–24] was labeled the refined integral method (RIM), involves simultaneously allowing the first and second residuals, over the moving domain, to vanish. It has been mentioned by several authors, for instance in [25–28], without proof that the RIM is equivalent to a first moment. By the kth residual resultant of the test profile $v(x,t)$, denote the multiple integral

$$\underbrace{\int_a^{\delta(t)} \int \cdots \iint_a^x}_{k} F\left(v(x,t), \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) (dx)^k$$

over the moving domain. Given that there exists a connection between the first moment and second residual, it is natural to seek to verify it and further examine if a similar and applicable relationship holds for arbitrary moments over the moving domain. The following simple but effective result applies.

Theorem 1. *Suppose the residuals of the state expression up to and including the kth vanish. Then the kth moment of the state expression, over the moving boundary, is a constant multiple of the $(k+1)$st residual.*

Proof. This proof follows by induction. Given a trial profile $v(x,t)$, we shall write $F\left(v(x,t), \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)$ as F in the following. It is clear that

$$\int_a^{\delta(t)} xF\,dx = -\int_a^{\delta(t)} \int_a^x F\,d\xi\,dx.$$

Likewise, the second moment which can be expressed as

$$\int_a^{\delta(t)} x^2 F\,dx = \delta^2(t) \int_a^{\delta(t)} F\,dx - 2\delta(t) \int_a^{\delta(t)} \int_a^x F\,d\xi\,dx + 2 \int_a^{\delta(t)} \int_a^x \int_a^\eta F\,d\eta\,d\xi\,dx \tag{4}$$

verifies the theorem when $k = 2$ since the first two integrals of the right hand side of Eq. (4) vanish. Suppose that the $(k-1)$ st moment is a constant multiple of the kth residual. Then, the proof of the theorem follows since

$$\int_a^{\delta(t)} x^k F\,dx = \delta(t) \int^{\delta(t)} x^{k-1} F\,dx - a \int^a x^{k-1} F\,dx - \int_a^{\delta(t)} \int^x \xi^{k-1} F\,d\xi\,dx = -\int_a^{\delta(t)} \int_a^x \xi^{k-1} F\,d\xi\,dx. \quad \square \tag{5}$$

A characterization of the RIM is the following corollary:

Corollary 1. *A solution of the defining ODE of the RIM is a zero of the first moment of the state expression.*