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Exact solution of the quadratic mixed-parity Helmholtz–Duffing oscillator

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ABSTRACT

In this paper, the exact solution of the quadratic mixed-parity Helmholtz–Duffing oscillator is derived by using Jacobi elliptic functions. It is also shown that the exact period of oscillation is given as a function of the complete elliptic integral of the first kind. At the end of the paper, we examine the stability of the system and determine the regions for which periodic and unbounded motions take place.

1. Introduction

This paper deals with the derivation of the exact solution of the mixed parity Helmholtz–Duffing oscillator of the form:

$$\ddot{x} + f(x) = 0, \quad f(x) = Ax + Bx^2 + \varepsilon x^3 + D_1, \tag{1.1}$$

where, x denotes the displacement of the system, A is the natural frequency, ε is a non-linear system parameter, and B and D_1 are system parameters independent of time. Notice that $f(x)$ in Eq. (1.1) is a mixed-parity function since the following conditions are satisfied [1]:

$$\begin{aligned} &f(x) = f_+(x) + f_-(x), \\ &f_+(-x) = f_+(x), \quad f_-(-x) = -f_-(x), \\ &f_+(x) \neq 0, \quad f_-(x) \neq 0. \end{aligned} \tag{1.2}$$

The purpose of this paper is to look at the exact solution of Eq. (1.1) by using Jacobian elliptic functions. The main motivation comes from the fact that exact solutions of certain equations of motion of the Duffing type are described precisely in terms of elliptic functions and its period is expressible in terms of a complete elliptic integral of the first kind [2]. Tamura [3], Rand [4], and Hu [5] used Jacobian elliptic functions to obtain the exact solution of a quadratic nonlinear oscillators. Also, the incomplete elliptic integral of the second kind was used by the author in [6] to developed the exact solution of Lame's equation.

Of course, when a closed-form solution of a nonlinear differential equation is unknown, we can use perturbation techniques to obtain its approximate solution. For instance, by using a slow space perturbative reduction, Hussein and Athel showed that there are two different special solutions of Eq. (1.1) with $\varepsilon = 0$, and $D_1 = 0$ that depend on the initial conditions to have stable or unstable system behavior [7]. Belhaq and Lakrad in [8] used the elliptic harmonic balance method to obtain the approximate solution of a strongly, mixed parity non-linear oscillator. Mickens in [9] studied the solution behaviors of three quadratic non-linear oscillators and concluded that there exist regions on which periodic solutions exist and that there are regions for which only unbounded motions take place. Hu in [10] used the method of harmonic balance to obtain solutions of quadratic nonlinear oscillators. He also used this method to calculate first order approximations to the periodic solution of Eq. (1.1) with $D_1 = 0$ and by imposing the restriction that initial conditions must be $x(0) = x_0 > 0$ and $\dot{x}(0) = 0$ [11]. Cao et al. established a function relationship between the symmetry breaking phenomenon and the symmetric parameter ε

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in the case for which $B = 1 - \varepsilon$ and $D_1 = 0$ in Eq. (1.1) [12]. They found that the symmetry breaking phenomena are strongly dependent on ε. Sun et al. study the non-linear response of strongly mixed parity nonlinear oscillators by modifying Lindstedt–Poincaré method [13]. They found highly accurate analytical approximate frequencies and the corresponding periodic solutions for small and large oscillation amplitudes. In an attempt to obtain approximate solutions to general strong single degree of freedom conservative systems, Sun and coworkers developed in [13] a new approach by introducing two odd nonlinear oscillators from the original nonlinear system described by Eq. (1.1) and obtained highly accurate results for both small an large amplitudes of oscillations.

From these previous works and references cited therein, it is evident that when the exact solution of a quadratic non-linear oscillator is unknown, perturbation techniques have to be used in an attempt to derive its approximate solution. Since the exact solution of Eq. (1.1) when $B = 0$ is based on Jacobi elliptic functions, we shall use elliptic functions and show that the exact solution of the mixed parity non-linear oscillator (1.1) is given in terms of a periodic rational form Jacobi elliptic function.

2. Exact solution

In order to obtain the exact solution of Eq. (1.1) by using Jacobian elliptic functions, we shall consider that $x(t)$ is given by the following equation:

$$x(t) = \frac{a - b + c(a+b)\mathrm{cn}(\omega t + \phi, k^2)}{1 + c\mathrm{cn}(\omega t + \phi, k^2)}, \tag{2.1}$$

where $cn(\omega t + \phi, k^2)$ is the cn Jacobian elliptic function that has a period in ωt equal to $4K(k^2)$, and $K(k^2)$ is the complete elliptic integral of the first kind for the modulus k, ω is the frequency of oscillation, $\phi, a, b,$ and c are unknown constant parameters that need to be determined. Notice that we have assumed that the exact solution of Eq. (1.1) has the rational form elliptic function (2.1), since Sarma and Rao in [14] and Mickens in [1] have derived accurate approximate periodic solutions to Eq. (1.1) by using rational harmonic balance approximations. Thus, substitution of Eq. (2.1) into Equation (1.1) gives:

$$\begin{aligned}
&aA - Ab + a^2B - 2abB + b^2B + a^3\varepsilon - 3a^2b\varepsilon + 3ab^2\varepsilon - b^3\varepsilon - 4bc^2\omega^2 + 4bc^2k^2\omega^2 + D_1 + \mathrm{cn}(\omega t + \phi, k^2) \\
&\quad \times \left[3aAc - Abc + 3a^2Bc - 2abBc - b^2Bc + 3a^3c\varepsilon - 3a^2bc\varepsilon - 3ab^2c\varepsilon + 3b^3c\varepsilon - 2bc\omega^2 + 4bck^2\omega^2 + 3cD_1\right] \\
&\quad + \mathrm{cn}(\omega t + \phi, k^2)^2\left[3aAc^2 + Abc^2 + 3a^2Bc^2 + 2abBc^2 - b^2Bc^2 + 3a^3c^2\varepsilon + 3a^2bc^2\varepsilon - 3ab^2c^2\varepsilon - 3b^3c^2\varepsilon\right. \\
&\quad \left. + 2bc^2\omega^2 - 4bc^2\omega^2 + 3c^2D_1\right] + \mathrm{cn}(\omega t + \phi, k^2)^3\left[aAc^3 + Abc^3 + a^2Bc^3 + 2abBc^3 + b^2Bc^3 + a^3c^3\varepsilon + 3a^2bc^3\varepsilon\right. \\
&\quad \left. + 3ab^2c^3\varepsilon + b^3c^3\varepsilon - 4bck^2\omega^2 + c^3D_1\right] = 0
\end{aligned} \tag{2.2}$$

in which the following identities for the $\mathrm{sn}(\omega t + \phi, k^2)$ and $\mathrm{dn}(\omega t + \phi, k^2)$ Jacobian elliptic functions:

$$\mathrm{sn}(\omega t + \phi, k^2)^2 + \mathrm{cn}(\omega t + \phi, k^2)^2 = 1; \quad \mathrm{dn}(\omega t + \phi, k^2)^2 + k^2\mathrm{sn}(\omega t + \phi, k^2)^2 = 1 \tag{2.3}$$

have been used.

Note that Eq. (2.2) hold for all time t if and only if:

$$aA - Ab + a^2B - 2abB + b^2B + a^3\varepsilon - 3a^2b\varepsilon + 3ab^2\varepsilon - b^3\varepsilon - 4bc^2\omega^2 + 4bc^2k^2\omega^2 + D_1 = 0, \tag{2.4}$$

$$\left[aAc^3 + Abc^3 + a^2Bc^3 + 2abBc^3 + b^2Bc^3 + a^3c^3\varepsilon + 3a^2bc^3\varepsilon + 3ab^2c^3\varepsilon + b^3c^3\varepsilon - 4bck^2\omega^2 + c^3D_1\right] = 0, \tag{2.5}$$

$$\left[3aAc - Abc + 3a^2Bc - 2abBc - b^2Bc + 3a^3c\varepsilon - 3a^2bc\varepsilon - 3ab^2c\varepsilon + 3b^3c\varepsilon - 2bc\omega^2 + 4bck^2\omega^2 + 3cD_1\right] = 0, \tag{2.6}$$

$$\left[3aAc^2 + Abc^2 + 3a^2Bc^2 + 2abBc^2 - b^2Bc^2 + 3a^3c^2\varepsilon + 3a^2bc^2\varepsilon - 3ab^2c^2\varepsilon - 3b^3c^2\varepsilon + 2bc^2\omega^2 - 4bc^2\omega^2 + 3c^2D_1\right] = 0. \tag{2.7}$$

Thus, we have four algebraic equations and six unknowns to say, a, b, c, k, ω, and ϕ. To determine the value of these parameters, we shall use the initial conditions of Eq. (1.1) that for convenience and without loss of generality, we assumed to be given as:

$$x(0) = x_{10}, \quad \dot{x}(0) = 0, \tag{2.8}$$

then from Eq. (2.1), $\phi = 0$ and

$$c = \frac{-a + b + x_{10}}{a + b - x_{10}}. \tag{2.9}$$