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S-asymptotically ω-periodic solutions to some classes of partial evolution equations

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ABSTRACT

In this paper, we give some sufficient conditions for the existence and uniqueness of S-asymptotically ω-periodic (mild) solutions to some classes of partial evolution equations in Banach spaces. The main result is obtained by means of the Banach fixed point principle.

1. Introduction

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space. In [1], the authors studied the existence and the uniqueness of almost automorphic solutions to the class of abstract partial evolution equations of the form

$$\frac{d}{dt}[u(t)+f(t,Bu(t))] = Au(t)+g(t,Cu(t)), \quad u(0)=0 \ t\in\mathbb{R}, \tag{1}$$

where A is the infinitesimal generator of an exponentially stable C_0-semigroup acting on $\mathbb{X}$; B, C are two densely defined closed linear operators on $\mathbb{X}$, and f, g are continuous functions.

The main purpose of our paper is to study the existence of S-asymptotically ω-periodic (mild) solutions to Eq. (1) assuming that f, g are S-asymptotically ω-periodic functions and $f(0,0) = 0$.

S-asymptotically ω-periodic functions constitute a class of functions larger than asymptotically ω-periodic ones. The literature relative to S-asymptotically ω-periodic functions remains limited due to the novelty of the concept. Qualitative properties of such functions are discussed in [3]. There are some papers dealing with the existence of S-asymptotically ω-periodic solutions of differential equations and fractional differential equations in finite as well as infinite dimensional spaces (cf. [2–6]). Of great interest is the paper by Lizama and N'Guérékata [5] where the authors created a chart establishing a general relationship between S-asymptotically ω-periodic functions and various subspaces of $BC(\mathbb{R},\mathbb{X})$. To the best of our knowledge, the problem treated here is new and our paper can inspire studies of many evolution equations (of fractional order as well) with S-asymptotically ω-periodic solutions.

We begin the work recalling some results on S-asymptotically ω-periodic functions. Then we apply these results to our problem. Theorem 1 is the main result of this paper.

2. Preliminaries

Let $\mathbb{X}$ be a Banach space. $BC(\mathbb{R}^+,\mathbb{X})$ denotes the space of the continuous bounded functions from $\mathbb{R}^+$ into $\mathbb{X}$; endowed with the norm $\|f\|_\infty := \sup_{t\geqslant 0}\|f(t)\|$, it is a Banach space. $C_0(\mathbb{R}^+,\mathbb{X})$ denotes the space of the continuous functions from $\mathbb{R}$ into $\mathbb{X}$

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such that $\lim_{t\to\infty} f(t) = 0$; it is a Banach subspace of $BC(\mathbb{R}^+, \mathbb{X})$. When we fix a positive number $\omega, P_\omega(\mathbb{X})$ denotes the space of all continuous ω-periodic functions from $\mathbb{R}^+$ into $\mathbb{X}$; it is a Banach subspace of $BC(\mathbb{R}^+, \mathbb{X})$ under the sup norm.

When $\mathbb{X}$ and $\mathbb{Y}$ are two Banach spaces, $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ denotes the space of the continuous linear mappings from $\mathbb{X}$ into $\mathbb{Y}$. If $\mathbb{X} = \mathbb{Y}$, we use the notation $\mathcal{L}(\mathbb{X})$ for $\mathcal{L}(\mathbb{X}, \mathbb{X})$.

Now we consider this set

$$C_0(\mathbb{R}^+, \mathbb{X}) := \{f \in BC(\mathbb{X}) : \lim_{t\to\infty} \|f(t)\| = 0\}.$$

Definition 1. Let $f \in BC(\mathbb{R}^+, \mathbb{X})$ and $\omega > 0$. We say that f is asymptotically ω-periodic if $f = g + h$ where $g \in P_\omega(\mathbb{X})$ and $h \in C_0(\mathbb{R}^+, \mathbb{X})$.

We denote by $AP_\omega(\mathbb{X})$ the set of all asymptotically ω-periodic functions from $\mathbb{R}^+$ to $\mathbb{X}$. It is a Banach space under the sup norm.

Note that we have

$$AP_\omega(\mathbb{X}) = P_\omega(\mathbb{X}) \oplus C_0(\mathbb{R}^+, \mathbb{X}).$$

Definition 2. A function $f \in BC(\mathbb{R}^+, \mathbb{X})$ is called S-asymptotically ω-periodic if there exists $\omega > 0$ such that $\lim_{t\to\infty}(f(t+\omega) - f(t)) = 0$. In this case we say that ω is an asymptotic period of f and that f is S-asymptotically ω-periodic.

We will denote by $SAP_\omega(\mathbb{X})$, the set of all S-asymptotically ω-periodic functions from $\mathbb{R}^+$ to $\mathbb{X}$. Then we have

$$AP_\omega(\mathbb{X}) \subset SAP_\omega(\mathbb{X}).$$

The inclusion is strict. Indeed consider the function $f : \mathbb{R}^+ \to c_0$ where $c_0 = \{x = (x_n)_{n\in\mathbb{N}} : \lim_{n\to\infty} x_n = 0\}$ equipped with the norm $\|x\| = \sup_{n\in\mathbb{N}} | x(n) |$, and $\left(f(t) = \frac{2nt^2}{t^2+n^2}\right)_{n\in\mathbb{N}}$. Then $f \in SAP_\omega(\mathbb{X})$ but $f \notin AP_\omega(\mathbb{X})$(see [3] Example 3.1).

The following result is due to Henriquez–Pierri–Tàboas; Proposition 3.5 in [3].

Theorem 1. *Endowed with the norm* $\|\cdot\|_\infty, SAP_\omega(\mathbb{X})$ *is a Banach space.*

Corollary 1 (*see [2], Corollary 3.10, p. 5*). *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be two Banach spaces, and let* $A \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$. *Then when* $f \in SAP_\omega(\mathbb{X})$, *we have* $Af := [t \to Af(t)] \in SAP_\omega(\mathbb{Y})$.

For the sequel we consider asymptotically ω-periodic functions with parameters.

Definition 3 (see [3]). A continuous function $f : [0, \infty[\times\mathbb{X} \to \mathbb{X}$ is said to be uniformly S-asymptotically ω-periodic on bounded sets if for every bounded set $K \subset \mathbb{X}$, the set $\{f(t,x) : t \geqslant 0,\ x \in K\}$ is bounded and $\lim_{t\to\infty}(f(t,x) - f(t+\omega,\ x)) = 0$ uniformly on $x \in K$.

Definition 4 (see [3]). A continuous function $f : [0, \infty[\times\mathbb{X} \to \mathbb{X}$ is said to be asymptotically uniformly continuous on bounded sets if for every $\epsilon > 0$ and every bounded set $K \subset \mathbb{x}$, there exist $L_{\epsilon,K} > 0$ and $\delta_{\epsilon,K} > 0$ such that $\|f(t,x) - f(t,y)\| < \epsilon$ for all $t \geqslant L_{\epsilon,K}$ and all $x, y \in K$ with $\|x - y\| < \delta_{\epsilon,K}$.

Theorem 2 (see [3]). *Let* $f : [0, \infty[\times\mathbb{X} \to \mathbb{X}$ *be a function which uniformly S-asymptotically* ω*-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. Let* $u : [0, \infty[$ *be S-asymptotically* ω*-periodic function. Then the Nemytskii operator* $\phi(\cdot) := f(\cdot, u(\cdot))$ *is S-asymptotically* ω*-periodic function.*

3. Main result

3.1. Preliminary results

Before the study of Eq. (1), we present some qualitative properties of S-asymptotically ω-periodic functions.

Proposition 1. *Let* $(\mathbb{X}, \|\cdot\|)$ *be a Banach space over the field* $\mathbb{K}$ *where* $\mathbb{K} = \mathbb{R}$, *or* $\mathbb{C}$. *If* $a(t) \in SAP_\omega(\mathbb{K})$ *and* $f(t) \in SAP_\omega(\mathbb{X})$, *then* $a(t)f(t) \in SAP_\omega(\mathbb{X})$.

Proof. Since $a(t)$ and $f(t)$ are bounded, $\exists M_1,\ M_2 \in \mathbb{R}+$ such that $| a(t) | \leqslant M_1$ and $\|f(t)\| \leqslant M_2,\ \forall t \geqslant 0$.

$$\begin{aligned}\lim_{t\to\infty} \|a(t+\omega)f(t+\omega) - a(t)f(t)\| &\leqslant \lim_{t\to\infty} \|(a(t+\omega) - a(t))f(t+\omega)\| + \lim_{t\to\infty} \|(f(t+\omega) - f(t))a(t)\| \\ &\leqslant \lim_{t\to\infty} \|(a(t+\omega) - a(t))\|M_2 + \lim_{t\to\infty} \|(f(t+\omega) - f(t))\|M_1 = 0. \qquad \square\end{aligned}$$