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Solution of Lane–Emden type equations using Legendre operational matrix of differentiation

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ABSTRACT

In the present paper, an efficient numerical method is developed for solving linear and nonlinear Lane–Emden type equations using Legendre operational matrix of differentiation. The proposed approach is different from other numerical techniques as it is based on differentiation matrix of Legendre polynomial. Some illustrative examples are given to demonstrate the efficiency and validity of the algorithm.

1. Introduction

Lane–Emden type equation models many phenomena in mathematical physics and astrophysics. It is a nonlinear differential equation which describes the equilibrium density distribution in self-gravitating sphere of polytrophic isothermal gas, has a singularity at the origin, and is of fundamental importance in the field of stellar structure, radiative cooling, and modeling of clusters of galaxies. The studies of singular initial value problems modeled by second order nonlinear ordinary differential equations (ODEs) have attracted many mathematicians and physicists. One of the equations in this category is the following Lane–Emden type equations:

$$y''(x) + \frac{\alpha}{x} y'(x) + f(x,y) = g(x), \quad \alpha, x \geqslant 0, \tag{1}$$

with initial conditions (IC)

$$y(0) = a, \quad y'(0) = 0, \tag{2}$$

where the prime denotes the differentiation with respect to x, a is constant, $f(x,y)$ is a nonlinear function of x and y. It is well known that an analytic solution of Lane–Emden type equation (1) is always possible [1] in the neighborhood of the singular point $x = 0$ for the above initial conditions. It is named after the astrophysists Jonathan H. Lane and Robert Emden, as it was first studied by them. Taking $\alpha = 2$, $f(x, y) = y^n$, $g(x) = 0$ and $a = 1$ in (1) and (2) respectively [2], we get

$$y''(x) + \frac{2}{x} y'(x) + y^n = 0, \quad x \geqslant 0, \tag{3}$$

which has another form,

$$\frac{1}{x^2} \frac{d}{dx}\left(x^2 \frac{dy}{dx}\right) + y^n = 0, \tag{4}$$

subject to IC

$$y(0) = 1, \quad y'(0) = 0. \tag{5}$$

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Classically, Eqs. (4) and (5) are known as the Lane–Emden equation. Similarly isothermal gas spheres [1] are modeled by

$$y'(x) + \frac{2}{x}y'(x) + e^{y(x)} = 0, \quad x \geqslant 0, \tag{6}$$

with IC

$$y(0) = 0, \quad y'(0) = 0. \tag{7}$$

The solutions of the Lane–Emden equation for a given index n are known as polytropes of index n. In (3), the parameter n has physical significance in the range $0 \leqslant n \leqslant 5$ and Eq. (3) with IC (5) has analytical solutions for $n = 0, 1, 5$ [3] and for other values of n, numerical solutions are sought. The series solution can be found by perturbation techniques and Adomian decomposition methods (ADM). However, these solutions are often, convergent in restricted regions. Thus, some techniques such as Pade's method is required to enlarge the convergent regions [1,4,5].

A number of algorithms have been proposed to solve (1) with $\alpha = 2$, $f(x,y) = f(y)$, a function of y alone and $g(x) = 0$. Some recent techniques are quasilineariziation method [6–8], a piecewise linearization technique [9] based on the piecewise linearization of the Lane–Emden equation and the analytic solution of the resulting piecewise constant coefficients ordinary differential equations, the homotopy analysis method (HAM) [10], and a variational approach using a semi-inverse method to obtain variational principle [11] and may employ the Ritz technique to obtain approximate solutions [12–14]. Later, Singh et al. [15], applied modified homotopy analysis method (MHAM) for the first time to obtain analytical approximate solution and showed that MHAM solution contains the previous solution obtained by ADM and HPM. Youseffi [16], has obtained the numerical solution of the Lane–Emden equation (1) by converting into an integral equation and then using Legendre wavelets, for $0 \leqslant x \leqslant 1$. Hybrid functions has been also used by Marzban et al. [17] to find out the numerical solution of (1) for some particular nonlinear cases in 2008.

In 2008, Dehghan and Shakeri [18] used the exponential transformation ($x = e^t$) with $\alpha = 2$, $f(x,y) = f(y)$ and $g(x) = 0$ to get

$$\ddot{y}(t) + \dot{y}(t) + e^{2t}f(y(t)) = 0, \tag{8}$$

subject to the conditions

$$\lim_{t\to-\infty} y(t) = a, \quad \lim_{t\to-\infty} e^{-t}\dot{y}(t) = 0, \tag{9}$$

where the symbol . denotes differentiation with respect to t and then applied variational iteration method (VIM) for the approximate solution. Pranand et al. [19,20] applied two different methods like rational Legendre pseudospectral approach and Hermite function collocation technique respectively to obtain an approximate solution. More recently, some more methods are also used to obtain the solution of Lane–Emden equations using perturbation techniques [21,22], optimal homotopy method [23] and Lagurre function collocation method [24].

The aim of the present paper is to propose a reliable numerical technique for solving linear and nonlinear Lane–Emden equation (1) using Legendre operational matrix of differentiation Saadatmandi and Dehghan [25]. Some special cases of the problem are solved to show its validity and efficiency as comparison with other existing numerical methods. The approximate solution obtained by the proposed method shows its superiority on the other existing numerical solutions [17–20].

2. Legendre polynomials and its operational matrix of Differentiation

The Legendre polynomials of order m defined by $L_m(t)$ are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formula:

$$L_0(t) = 1, \quad L_1(t) = t \quad L_{m+1}(t) = \frac{2m+1}{m+1}tL_{m-1}(t) - \frac{m}{m+1}L_{m-1}(t), \quad m = 1,2,\ldots \tag{10}$$

These polynomials on the interval $t \in [0,1]$ so called shifted Legendre polynomials can be defined by introducing the change of variable $t = 2x - 1$. Let the shifted Legendre polynomials $L_m(2x-1)$ be denoted by $P_m(x)$. Then $P_m(x)$ can be obtained as follows:

$$P_{m+1}(x) = \frac{(2m+1)(2x-1)}{m+1}P_m(x) - \frac{m}{m+1}P_{m-1}(x), \quad m = 1,2,\ldots \tag{11}$$

where $P_0(x) = 1$ and $P_1(x) = 2x - 1$. The analytic form of the shifted Legendre polynomials $P_m(x)$ of degree m are given by:

$$P_m(x) = \sum_{i=0}^{m}(-1)^{m+i}\frac{(m+i)!x^i}{(m-1)(i!)^2}. \tag{12}$$

Any function, $y(x) \in L^2[0,1]$, can be approximated as a sum of shifted Legendre polynomials as:

$$y(x) = \sum_{i=0}^{\infty} c_i P_i(x), \tag{13}$$

where