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Identities involving generalized Fibonacci-type polynomials ☆

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ABSTRACT

In this paper we consider a family of generalized Fibonacci-type polynomials. These polynomials have a lot of similar properties to the generalized Jacobsthal-type polynomials. As an extension of the work of Djordjević [G.B. Djordjević, Mixed convolutions of the Jacobsthal type, Appl. Math. Comput. 186 (2007) 646–651], we give some recurrence relations and identities involving the generalized Fibonacci-type polynomials.

1. Introduction

The Fibonacci sequence $\{F_n\}$ is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geqslant 2,$$

with initial conditions $F_0 = 0$ and $F_1 = 1$. There is a tradition of using polynomials to study Fibonacci numbers. Although Fibonacci polynomials have been well studied, there was initially no common agreement on how to define this class of polynomials. For example, Catalan [2] defined them by

$$f_0(x) = 0, \quad f_1(x) = 1, \quad f_{n+1}(x) = xf_n(x) + f_{n-1}(x),$$

while Jacobsthal [8] defined them by

$$f_0(x) = 0, \quad f_1(x) = 1, \quad f_{n+1}(x) = f_n(x) + xf_{n-1}(x).$$

In [10], Schwerdtfeger presented a family of matrices

$$B = \begin{pmatrix} 1 & b \\ 1 & 0 \end{pmatrix}.$$

By induction, it can be checked that

$$B^n = \begin{pmatrix} f_n(b) & bf_{n-1}(b) \\ f_{n-1}(b) & bf_{n-2}(b) \end{pmatrix} \quad \text{for } n \geqslant 1,$$

where the Fibonacci polynomials are defined by $f_{n+1}(x) = f_n(x) + xf_{n-1}(x)$ with $f_{-1} = 0$ and $f_0 = 1$. Clearly, we have $f_n(1) = F_{n+1}$. In [11], Silvester showed that a number of the properties of the Fibonacci sequence can be derived from a matrix representation. From then on, there has been an amount of research devoted to this topic (see [6,9] for instance).

Let $m \geqslant 2$ be a fixed positive integer. We will consider two classes of polynomials, i.e., *the generalized Fibonacci polynomials* $\{G_{n,m}(x)\}$ and *the generalized Lucas polynomials* $\{g_{n,m}(x)\}$, defined, respectively, by

$$G_{n+m,m}(x) = G_{n+m-1,m}(x) + xG_{n,m}(x), \quad n \geqslant 0, \tag{1}$$

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with initial conditions $G_{0,m}(x) = 0,\ G_{n,m}(x) = 1,\ n = 1, 2, \ldots, m-1$ and

$$g_{n+m,m}(x) = g_{n+m-1,m}(x) + xg_{n,m}(x), \quad n \geqslant 0, \tag{2}$$

with initial conditions $g_{0,m}(x) = 2,\ g_{n,m}(x) = 1,\ n = 1, 2, \ldots, m-1$. In this paper we call these polynomials *the generalized Fibonacci-type polynomials*.

The polynomials $G_{n,2}(2x)$ and $g_{n,2}(2x)$ were considered in [7]. The polynomials $G_{n,m}(2x)$ and $g_{n,m}(2x)$ are respectively considered to be the generalized Jacobsthal-type polynomials $J_{n,m}(x)$ and $j_{n,m}(x)$, which were studied by Djordjević [4,5]. The generalized Fibonacci numbers $G_{n,m}(1)$ were studied in [1]. In particular, when $m = 2$ and $x = 1$, the numbers $G_{n,2}(1)$ and $g_{n,2}(1)$ are respectively reduced to the Fibonacci numbers F_n and the Lucas numbers L_n.

This paper is an extension of the work of Djordjević [4,5]. The organization of the paper is as follows. In the next section, we provide several recurrence relations for the generalized Fibonacci-type polynomials. In section 3, we prove some identities involving these polynomials by matrix method.

2. Recurrence relations

Here are the first few polynomials when $m = 4$:

$$\begin{aligned}
&G_{0,4}(x) = 0, \quad G_{1,4}(x) = G_{2,4}(x) = G_{3,4}(x) = G_{4,4}(x) = 1, \quad G_{5,4}(x) = 1 + x,\\
&G_{6,4}(x) = 1 + 2x, \quad G_{7,4}(x) = 1 + 3x, \quad G_{8,4}(x) = 1 + 4x, \quad G_{9,4}(x) = 1 + 5x + x^2;\\
&g_{0,4}(x) = 2, \quad g_{1,4}(x) = g_{2,4}(x) = g_{3,4}(x) = 1, \quad g_{4,4}(x) = 1 + 2x, \quad g_{5,4}(x) = 1 + 3x,\\
&g_{6,4}(x) = 1 + 4x, \quad g_{7,4}(x) = 1 + 5x, \quad g_{8,4}(x) = 1 + 6x + 2x^2, \quad g_{9,4}(x) = 1 + 7x + 5x^2.
\end{aligned}$$

There are several properties that we can observe here. For example, look at the degree of the polynomials, we have $\deg G_{n,m}(x) = \lfloor \frac{n-1}{m} \rfloor$ and $\deg g_{n,m}(x) = \lfloor \frac{n}{m} \rfloor$. Now, we give a relation between these polynomials.

Proposition 1 [3, (1.10)]. *For $n \geqslant m-1$, we have*

$$g_{n,m}(x) = G_{n+1,m}(x) + xG_{n-m+1,m}(x).$$

Proof. By Binet's Formula [7], it is easy to verify that the result is true for $m = 2$. We will consider the case $m \geqslant 3$. We proceed by induction on n. Note that $g_{2,m}(x) = 1 = 1 + 0 = G_{3,m}(x) + xG_{3-m,m}(x)$, where the last equality comes from the definition of the first few terms of (1). So the statement is true for $n = 2$. Assume that the statement is true for $n = k$. Then

$$\begin{aligned}
g_{k+1,m}(x) &= g_{k,m}(x) + xg_{k-m+1,m}(x) = G_{k+1,m}(x) + xG_{k-m+1,m}(x) + xG_{k-m+2,m}(x) + x^2G_{k-2m+2,m}(x)\\
&= (G_{k+1,m}(x) + xG_{k-m+2,m}(x)) + x(G_{k-m+1,m}(x) + xG_{k-2m+2,m}(x)) = G_{k+2,m}(x) + xG_{k-m+2.m}(x).
\end{aligned}$$

Thus the statement is true for $k+1$, as desired. $\square$

From (1) and (2), it is easy to verify that

$$\begin{aligned}
G(x,t) &= \sum_{n \geqslant 0} G_{n,m}(x)t^n = \frac{t}{1 - t - xt^m},\\
g(x,t) &= \sum_{n \geqslant 0} g_{n,m}(x)t^n = \frac{2 - t}{1 - t - xt^m}.
\end{aligned}$$

Explicitly, for $n \geqslant m$,

$$G_{n,m}(x) = \sum_{k=0}^{\lfloor \frac{n-1}{m} \rfloor} \binom{n-1-k(m-1)}{k} x^k,$$

and

$$g_{n,m}(x) = \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} \frac{n-(m-2)k}{n-(m-1)k} \binom{n-k(m-1)}{k} x^k.$$

We can now present the first main result of this paper.

Theorem 2. *The polynomials $G_{n,m}(x)$ and $g_{n,m}(x)$ satisfy the following recurrence relations*:

(c_1) $x(G_{0,m}(x) + G_{1,m}(x) + G_{2,m}(x) + \cdots + G_{n,m}(x)) = G_{n+m,m}(x) - 1$;
(c_2) $G_{i,m}(x)x^n + G_{m+i,m}(x)x^{n-1} + G_{2m+i,m}(x)x^{n-2} + \cdots + G_{nm+i,m}(x) = G_{nm+i+1,m}(x)$, *for* $i = 1, 2, \ldots, m-1$;
(c_3) $G_{m,m}(x)x^{n-1} + G_{2m,m}(x)x^{n-2} + G_{3m,m}(x)x^{n-2} + \cdots + G_{nm,m}(x) = G_{nm+1,m}(x) - x^n$, *for* $n \geqslant 1$;