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Anisotropic error bounds of Lagrange interpolation with any order in two and three dimensions ☆

Shaochun Chen [a], Yanjun Zheng [a], Shipeng Mao [b,*]

[a] Department of Mathematics, Zhengzhou University, Zhengzhou 450052, China

[b] Institute of Computational Mathematics and Science/Engineering Computing, Academy of Mathematics and System Science, Chinese Academy of Sciences, P.O. Box: 2719, Beijing 100190, China

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ABSTRACT

In this paper, using the Newton's formula of Lagrange interpolation, we present a new proof of the anisotropic error bounds for Lagrange interpolation of any order on the triangle, rectangle, tetrahedron and cube in a unified way.

1. Introduction

It is known that the polynomial interpolations are the foundations of construction the finite elements and the interpolation error estimates play a key role in deriving a priori error estimates of the finite element methods. The main strategy of the traditional interpolation theory is fairly standard, namely, first deriving the estimate on the reference element and then an application of a coordinate transformation between a general element and the reference element, see [11,7] and references therein. For the triangular and rectangular elements in two dimension and the tetrahedral and cubic elements in three dimension, the mapping between a general element and the reference element is an affine mapping, so in the following we call these elements affine elements. The classical error estimates of the polynomial interpolation on the affine elements need the regular [11] or nondegenerate [7] condition, i.e., the ratio of the diameters of the element and the biggest ball contained in the element is uniformly bounded. This condition restricts the applications of the finite elements. It is found (see e.g., [6,15]) a long time ago that this condition is not necessary for some interpolation error estimates. We call the element does not satisfy the regular condition the anisotropic element. Recently, the research of the anisotropic elements is rapidly developed, and there are several different methods dealing with them. Apel and Dobrowolski [3], Apel [4] gave one anisotropic form of the interpolation error on the reference element. They got the anisotropic interpolation error estimates on a general element for some Lagrange and Hermite elements under the maximal angle and coordinate system conditions. The corresponding appeared derivatives are along the coordinate directions. Chen et al. [9,10] extend this method by presenting a simple anisotropic criterion on the reference element and analyzed some nonconforming elements. Acosta [1], Acosta and Duran [2], Duran [12,13] got the anisotropic error estimates for low order Lagrange and R-T interpolations by using of the average property of the interpolation and the appeared derivatives under consideration are along the directions of the element boundary. The different forms of the anisotropic error estimate of the linear triangular Lagrange interpolation are obtained by the decomposition of the transformation matrix between a general element and the reference element in [14] and by Taylor's expansion in [8].

In this paper, the anisotropic interpolation error estimates of Lagrange interpolations with any order on the affine elements (triangle, rectangle, cube and tetrahedron) are derived in a unified new way. On the reference element the anisotropic error

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* Corresponding author.
E-mail addresses: shchchen@zzu.edu.cn (S. Chen), zhengyanjun@zzu.edu.cn (Y. Zheng), maosp@lsec.cc.ac.cn (S. Mao).

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estimates of the interpolations are proved by Newton's formula of the Lagrange interpolation and a special property of the divided difference, which are different from [4]. The appeared derivatives are along the directions of the element boundary (as in [2,13]) and independent length scales in different directions are extracted (as in [4]). No geometry condition of the element is needed for rectangular and cubic elements. The sine of the biggest internal angle of the element and the regular vertex property factor [2] appear explicitly in the triangular and the tetrahedral elements, respectively, then standard arguments will lead to the estimates that depend on the biggest internal angle of the element and the regular vertex property factor.

2. Lagrange interpolation remainder term on reference elements

2.1. The property of the divided difference

Let $x_0 < x_1 < \cdots < x_m$ be a uniform partition, $d = x_{i+1} - x_i$, $0 \leqslant i \leqslant m-1$.
It is easy to get the following result by inductive method.

Lemma 2.1

$$\int_{x_1}^{x_2} dt_1 \int_{t_1}^{t_1+d} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+d} g(t_m)dt_m = \int_{x_0}^{x_1} dt_1 \int_{t_1}^{t_1+d} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+d} g(t_m+d)dt_m. \tag{2.1}$$

Let $f[x_0,\ldots,x_m]$ be the usual divided difference (see [5]), then we get the following lemma.

Lemma 2.2. *Suppose f(x) is sufficiently smooth, then:*

$$f[x_0,\ldots,x_m] = \frac{1}{m!d^m} \int_{x_0}^{x_1} dt_1 \int_{t_1}^{t_1+d} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+d} f^{(m)}(t_m)dt_m. \tag{2.2}$$

Proof. We use the inductive method.
When $m = 1, f[x_0,x_1] = \frac{1}{d}\int_{x_0}^{x_1} f'(t_1)dt_1$, (2.2) is evident.
Suppose (2.2) holds for any $m \geqslant 1$, then:

$$\begin{aligned} f[x_0,\ldots,x_{m+1}] &= (f[x_1,\cdots,x_{m+1}] - f[x_0,\ldots,x_m])/(x_{m+1}-x_0) \\ &= \frac{1}{(m+1)d}\left[\int_{x_1}^{x_2} dt_1 \int_{t_1}^{t_1+d} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+d} f^{(m)}(t_m)dt_m - \int_{x_0}^{x_1} dt_1 \int_{t_1}^{t_1+d} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+d} f^{(m)}(t_m)dt_m\right] \Big/ (m!d^m) \\ &\overset{(2.1)}{=} \frac{1}{(m+1)!d^{m+1}} \int_{x_0}^{x_1} dt_1 \int_{t_1}^{t_1+d} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+d} \left[f^{(m)}(t_m+d) - f^{(m)}(t_m)\right]dt_m \\ &= \frac{1}{(m+1)!d^{m+1}} \int_{x_0}^{x_1} dt_1 \int_{t_1}^{t_1+d} dt_2 \cdots \int_{t_{m-1}}^{t_{m-1}+d} dt_m \int_{t_m}^{t_m+d} f^{(m+1)}(t_{m+1})dt_{m+1}. \end{aligned}$$

This completes the proof. □

Remark 1. Lemma 2.2 is similar to Hermite–Gennochi Theorem 5, Theorem 3.3.
Using the inductive method again, we can get:

Lemma 2.3. *For all $0 \leqslant l \leqslant m$, $f[x_0,\ldots,x_m]$ can be expressed by*

$$f[x_0,\ldots,x_m] = \sum_{i=0}^{m-l} c_i f[x_i,\ldots,x_{i+l}], \tag{2.3}$$

where c_i $(0 \leqslant i \leqslant m-l)$ is only dependent on l and d.

The interpolation polynomial $If(x)$ of $f(x)$ satisfying $If(x_i) = f(x_i)(0 \leqslant i \leqslant m)$ can be expressed in the following two forms, where (2.4) is called Lagrange's formula and (2.5) is called Newton's formula (see [5]):

$$If(x) = \sum_{i=0}^{m} f(x_i)p_i(x), \tag{2.4}$$

where $p_i(x)$ $(0 \leqslant i \leqslant m) \in P_m$ (the polynomial space of degree less or equal to m) and $p_i(x_j) = \delta_{ij}$, $0 \leqslant i, j \leqslant m$:

$$If(x) = \sum_{i=0}^{m} f[x_0,\ldots,x_i] \prod_{j=0}^{i-1}(x-x_j). \tag{2.5}$$