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Backward doubly stochastic differential equations with weak assumptions on the coefficients

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A B S T R A C T

In this paper, we deal with one dimensional backward doubly stochastic differential equations (BDSDEs). We obtain existence theorems and comparison theorems for solutions of BDSDEs with weak assumptions on the coefficients.

1. Introduction

Pardoux and Peng [14] introduced the following nonlinear backward stochastic differential equations (BSDEs):

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T].$$

They obtained the existence and uniqueness of solutions under the Lipschitz condition. Since then, the theory of BSDEs has been developed by many researchers and there are many works attempting to weaken the Lipschitz condition in order to obtain the existence and uniqueness results of BDSDEs (see e.g., Bahlali [1], Briand and Confortola [3], Darling and Pardoux [5], El Karoui and Huang [6], Hamadène [7], Jia [8], Kobylanski [9], Lepeltier and San Martin [10] and the references therein). Today the BSDE has become a powerful tool in the study of partial differential equations, risk measures, mathematical finance, as well as stochastic optimal controls and stochastic differential games.

After the nonlinear BSDEs were introduced, Pardoux and Peng [15] brought forward BDSDEs with two different directions of stochastic integrals, i.e., the equations involve both a standard stochastic Itô's integral and a backward stochastic Itô's integral:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T Z_s dW_s, \quad t \in [0, T], \tag{1.1}$$

the integral with respect to $\{B_t\}$ is a backward Itô's integral and the integral with respect to $\{W_t\}$ is a standard forward Itô's integral. By virtue of this kind of BDSDE, Pardoux and Peng [15] established the connections between certain quasi-linear stochastic partial differential equations and BDSDEs, and obtained a probabilistic representation for a class of quasi-linear stochastic partial differential equations. They established the existence and uniqueness results for solutions of BDSDEs under the Lipschitz condition on the coefficients. This kind of BDSDEs has a practical background in finance. The extra noise B can be regarded as some extra information, which can not be detected in the financial market, but is available to the particular investors.

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Since the work of Pardoux and Peng [15], there are only several works attempting to relax the Lipschitz condition to get the existence and uniqueness results for one dimensional BDSDEs. Shi et al. [16] obtained that one dimensional BDSDE (1.1) has at least one solution if f is continuous and of linear growth in (y,z), and $\{f(t,0,0)\}_{t\in[0,T]}$ is bounded. Under the assumptions that f is bounded, left continuous and non-decreasing in y and Lipschitz in z, Lin [11] established an existence theorem for one dimensional BDSDE (1.1). Lin [12] proved that one dimensional BDSDE (1.1) has at least one solution if the coefficient f is left Lipschitz and left continuous in y, and Lipschitz in z. Lin and Wu [13] obtained a uniqueness result for one dimensional BDSDE (1.1) under the conditions that f is Lipschitz in y and uniformly continuous in z.

Motivated by the above results, one of the objectives of this paper is to get an existence theorem for one dimensional BDSDE (1.1), which generalizes the result in Shi et al. [16] by the condition of the square integrability of $\{f(t,0,0)\}_{t\in[0,T]}$ instead of the boundedness of $\{f(t,0,0)\}_{t\in[0,T]}$. The other objective of this paper is to generalize the existence result in Lin [12]. We consider the following BDSDE:

$$Y_t = \xi + \int_t^T \left(sgn(Y_s)Y_s^2 + \sqrt{Z_s 1_{\{Z_s\geqslant 0\}}}\right)ds + \int_t^T g(s,Y_s,Z_s)dB_s - \int_t^T Z_s dW_s, \quad t\in[0,T].$$

Since $\sqrt{z1_{\{z\geqslant 0\}}}$ is not Lipschitz in z, then we can not apply the existence result in Lin [12] to get the existence theorem of the above BDSDE. We shall investigate an existence result for one dimensional BDSDE (1.1) where f is left Lipschitz and left continuous in y and uniformly continuous in z, which improves the result in Lin [12]. Since f is uniformly continuous in z, then we can not apply comparison theorems for solutions of BDSDEs in [16,12]. In order to get the existence theorem for solutions of BDSDEs we shall first establish a comparison theorem for solutions of BDSDEs when f is Lipschitz in y and uniformly continuous in z, which plays an important role.

This paper is organized as follows: In Section 2, we give some preliminaries and notations, which will be useful in what follows. In Section 3, we obtain an existence theorem for the solutions of BDSDEs with continuous coefficients. In Section 4, we establish an existence theorem and a comparison theorem for the solutions of a class of BDSDEs with discontinuous coefficients.

2. Preliminaries and notations

Let $T > 0$ be a fixed terminal time and $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. Let $\{W_t\}_{0\leqslant t\leqslant T}$ and $\{B_t\}_{0\leqslant t\leqslant T}$ be two mutually independent standard Brownian motion processes, with values in $\mathbb{R}^d$ and $\mathbb{R}^l$, respectively, defined on $(\Omega,\mathcal{F},\mathbb{P})$. Let $\mathcal{N}$ denote the class of $\mathcal{P}$-null sets of $\mathcal{F}$. Then, we define

$$\mathcal{F}_t \doteq \mathcal{F}_{0,t}^W \vee \mathcal{F}_{t,T}^B, \quad t\in[0,T],$$

where for any process $\{\eta_t\}$, $\mathcal{F}_{s,t}^\eta = \sigma\{\eta_r-\eta_s, s\leqslant r\leqslant t\}\vee\mathcal{N}$. Let us point out that $\mathcal{F}_{0,t}^W$ is increasing and $\mathcal{F}_{t,T}^B$ is decreasing in t, but $\mathcal{F}_t$ is neither increasing nor decreasing in t. Let us introduce the following spaces:

- $L^2(\Omega,\mathcal{F}_T,\mathbb{P})\doteq\{\xi : \mathcal{F}_T$-measurable random variable such that $\mathbb{E}[|\xi|^2]<\infty\}$.
- $S^2(0,T;\mathbb{R})\doteq\{\varphi : \varphi$ is a continuous process with value in $\mathbb{R}$ such that $\|\varphi\|_{S^2}^2 = \mathbb{E}[\sup_{0\leqslant t\leqslant T}|\varphi_t|^2]<\infty$, and φ_t is $\mathcal{F}_t$-measurable, for all $t\in[0,T]\}$.
- $M^2(0,T;\mathbb{R}^d)\doteq\{\varphi : \varphi$ is a jointly measurable process with value in $\mathbb{R}^d$ such that $\|\varphi\|_{M^2}^2 = \mathbb{E}[\int_0^T|\varphi_t|^2dt]<\infty$, and φ_t is $\mathcal{F}_t$-measurable, for all $t\in[0,T]\}$.

Let

$$g:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^l.$$

In this paper, we suppose that $\xi\in L^2(\Omega,\mathcal{F}_T,\mathbb{P})$ and g always satisfies the following assumptions:

$(H1)$ (Lipschitz condition): There exist constants $C > 0$ and $0 < \alpha < 1$ such that, for all $(t,y_i,z_i)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d$, i = 1, 2,

$$|g(t,y_1,z_1)-g(t,y_2,z_2)|^2\leqslant C|y_1-y_2|^2+\alpha|z_1-z_2|^2.$$

$(H2)$ For all $(y,z)\in\mathbb{R}\times\mathbb{R}^d$, $g(\cdot,y,z)\in M^2(0,T;\mathbb{R}^l)$.

Let

$$f:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$$

be such that, for all $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d$, $f(t,y,z)$ is $\mathcal{F}_t$-measurable. We make the following assumptions:

$(H3)$ (Lipschitz condition): There exists a constant $C > 0$ such that, for all $(t,y_i,z_i)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d$, i = 1, 2,

$$|f(t,y_1,z_1)-f(t,y_2,z_2)|\leqslant C(|y_1-y_2|+|z_1-z_2|).$$

$(H4)$ $f(\cdot,0,0)\in M^2(0,T;\mathbb{R})$.