Applied Mathematics and Computation 217 (2011) 9451–9466

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On self-similar solutions of semilinear wave equations in higher space dimensions

Radosław Kycia

Institute of Physics, Jagiellonian University, Kraków, Poland

ARTICLE INFO

Keywords:
Self-similar solution
Semilinear wave equation
Singular boundary value problem

ABSTRACT

In this paper we analyze self-similar solutions of the semilinear wave equation $\Phi_{tt} - \Delta\Phi - \Phi^p = 0$ for $n > 3$ space dimensions. We found several classes of analytic solutions labeled by a single parameter, the form of which differ in the vicinity of the light cone. We also propose suitable numerical methods to study them.

1. Introduction

In many nonlinear PDEs blowup occurs, i.e., singularity is formed in finite time in evolution starting with smooth initial conditions. Blowup is almost always connected with an interesting physical phenomenon (see [1] and references therein) and it is often well described in terms of self-similarity. Spherically symmetric self-similar solutions are defined by

$$\Phi(t,r) = (T-t)^{-\alpha}u(\rho), \quad \rho = \frac{r}{T-t}, \tag{1}$$

where $r = |x|$ is the radius. This is a special type of the general self-similar solutions $\Phi_\lambda = \lambda^{-\alpha}\,\Phi(t/\lambda, x/\lambda)$. Assuming that the similarity profile $u(\rho)$ is analytic, we obtain blowup when $t \to T$, therefore, it is crucial to have the detailed description of self-similar profiles.

The example of PDE where blowup occurs is the semilinear wave equation with power nonlinearity

$$\Phi_{tt} - \Delta\Phi - \Phi^p = 0, \quad \Phi = \Phi(x,t), \quad x \in R^n, \tag{2}$$

which will be examined in this paper. Solutions depend on two parameters p and n, which are integer numbers by assumption. In this paper we limit ourselves to $n > 3$ and odd $p > 2$. If p is even, the nonlinear term Φ^p should be replaced by $|\Phi|^{p-1}\Phi$ to keep the reflection symmetry for Φ.

The Eq. (2) has the energy functional in the form [7]

$$E[\Phi] = \int_{\mathbf{R}^n} \left(\Phi_t^2 + (\nabla\Phi)^2 - \frac{1}{p+1}\Phi^{p+1} \right) d^n x, \tag{3}$$

which scales on the general self-similar solutions as $E[\Phi_\lambda] = \lambda^\beta E[\Phi]$, where $\beta = \frac{(n-2)p-(n+2)}{p-1}$. In general, scaling is called subcritical for $\beta < 0$, supercritical for $\beta > 0$ and critical when $\beta = 0$. In our case the scaling is critical when

$$p = p_Q = \frac{n+2}{n-2}. \tag{4}$$

E-mail address: radoslaw.kycia@uj.edu.pl

0096-3003/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2011.04.039

There is a vast literature that considers various aspects related to the Eq. (2) and its generalizations, e.g., [2–8]; for scattering method approach see, e.g., [9]. Strichartz estimates and their applications are given in [10,11]. The application of Besov space method is presented in [12,13]. We encourage the interested reader to study them. We will use (2) only as a starting point in the search of its self-similar solutions.

It is well known that the Eq. (2) has two particular solutions [14]. The first one is obtained when we neglect coordinate dependence and solve the resulting ODE

$$\Phi_0(t) = \frac{b_0}{(T-t)^{\alpha}}, \quad b_0 = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}, \quad \alpha = \frac{2}{p-1}, \quad T > 0 \tag{5}$$

with the corresponding constant profile

$$u_0(\rho) = b_0. \tag{6}$$

The second one is the static spherically symmetric solution of the form

$$\Phi_\infty(r) = b_\infty r^{-\alpha}, \quad b_\infty = \left(\frac{2(p(n-2)-n)}{(p-1)^2}\right)^{\frac{1}{p-1}} \tag{7}$$

and the associated self-similar profile

$$u_\infty(\rho) = b_\infty \rho^{-\alpha} \tag{8}$$

is unbounded when ρ tends to 0. The other self-similar profiles (we will also use the name self-similar solutions for profiles) are obtained from second order ordinary differential equation for similarity profile $\left({}' = \frac{d}{d\rho}\right)$

$$(1-\rho^2)u'' + \left(\frac{n-1}{\rho} - \frac{2(p+1)}{p-1}\rho\right)u' - \frac{2(p+1)}{(p-1)^2}u + u^p = 0, \tag{9}$$

which results from substituting (1) into (2). Casual structure and finite speed of propagation introduced by (2) makes the blowup at the point $(t = T, r = 0)$ only connected with its past light cone, which corresponds to the interval $\rho \in [0;1]$. This is a main reason to investigate the existence of global analytic self-similar profiles of (9) on this interval. That solutions of (9), if exist, connect two singular points $\rho = 0$ and $\rho = 1$ of this equation along analytic curve.

The paper is organized as follows. Starting from (9), we will study its analytic solutions. We begin from local existence theorems at $\rho = 0$ and $\rho = 1$ in Section 2. Asymptotics at $\rho = 0$ is a generalization of the results from [14], but at $\rho = 1$ we obtain several new classes of solution, which depend on a value of n, p and differ from each other in analytical form. Then in Section 3, generalizing [14], we try to match these two asymptotics to obtain an analytic solution in the entire interval $\rho \in [0;1]$. This matching is only possible for some special initial data at both endpoints and values of n, p; the requirement of smooth matching is a sort of quantization condition for the values of solution parameters. Solutions with these quantized parameters obey remarkable scaling laws derived in Section 4. Apart form those known in the $n = 3$ case we observe also qualitatively new shapes of this scaling. We also propose a special numerical method which is best suitable to explain the global existence for one class of new-found solutions. In our studies we will be using mainly asymptotic methods (see, e.g., [15,16]).

2. Local existence

In this section we examine local analytic solutions of (9) at both endpoints of $[0;1]$. We will be searching for power series solutions which will be easily obtained by employing the Cauchy product [17]

$$\begin{gathered}\left(\sum_{l=0}^{\infty} a_l (x-x_0)^l\right)^p = \sum_{l=0}^{\infty} c_l (x-x_0)^l, \\ c_0 = a_0^p, \quad c_m = \frac{1}{m a_0}\sum_{l=1}^{m}(lp - m + l)a_l c_{m-l}\end{gathered} \tag{10}$$

for $m > 0$, which simplifies nonlinear term in (9).

2.1. Local solution at $\rho = 0$

We start with the construction of the solution around $\rho = 0$. Substituting the formal power series $u(\rho) = \sum_{l=0}^{\infty} a_l \rho^l$ into (9) and using (10) we get

$$\sum_{l=-2}^{\infty}(l+2)(l+n)a_{l+2}\rho^l = \sum_{l=0}^{\infty}[l(l-1) + l(2\alpha+2) + \alpha(\alpha+1)a_l - c_l]\rho^l, \tag{11}$$