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Iterative solution of some nonlinear differential equations

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ABSTRACT

We propose an iterative method to solve some non-linear ordinary differential equations. Comparing on the Mathieu, van der Pol and Hill equation of fourth order, we see that this method is much more efficient than the well known methods by Lyapunov or Picard.

1. Introduction

We intend to determine explicit approximate solutions, with given initial values, of equations of the type

$$z''(x) + \omega^2 z(x) = f(x,z,z'), \quad z(x_a) = \alpha, \quad z'(x_a) = \beta. \tag{1}$$

In particular, we shall focus our attention in the search for periodic solutions. There exists several perturbative methods to determine explicit approximate solutions. Most of them require a small perturbative parameter. In this context, we can mention for instance the Lindstedt–Poincaré, Krylov–Bogolubov–Mitropolskii perturbation methods and also the multi-time expansion method [1]. When a perturbative parameter cannot be found, one can use the harmonic balance to obtain periodic approximations [2,3]. The need of efficient ways to find approximate solutions in the case of strongly nonlinear equations has been stressed in [4–6]. Apart from those mentioned general methods, there exist a more specific approach for some equations like the van der Pol equation [7–12]. Different iteration schemes are discussed in [13,14]. The implementation of all these methods of resolution have a certain degree of difficulty that can be somehow overcome with the use of a software like Mathematica. In this article, we present an effective calculation tool, which is conceptually quite simple and is inspired in the method of successive approximations of Picard–Liouville [12–15].

In this kind of articles in which an analytic solution is search using iterations a study of the uniform convergence of the approximate solutions to the exact solution is an absolute requirement. This result is shown in Appendix A.

This paper is organized as follows: In Section 2, we propose our iterative method. Sections 3–5 are devoted to applications to given linear (Mathieu or Hill of fourth order) or non-linear (van der Pol) differential equations. In addition, we compare our results with those obtained with Lyapunov and Picard methods. Finally, in Appendix A, we discuss the convergence of our method.

2. The iterative method

We shall obtain explicit solutions of Eq. (1), where f is a continuous function of three variables continuous on a domain that includes the range $x_a \leqslant x \leqslant x_b$ for the first variable. Inspired in the Picard method [12,15], we try to solve Eq. (1) by iteration. At the kth step (1) looks like

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$$\begin{aligned} &z''_{k+1}(x)+\omega^2 z_{k+1}(x)=f(x,z_k,z'_k),\\ &x_a\leqslant x\leqslant x_b,\quad z_{k+1}(x_a)=\alpha,\quad z'_{k+1}(x_a)=\beta,\quad k=1,2,\ldots, \end{aligned} \tag{2}$$

where $z_0(x)$ is an initial test function and α and β given real numbers. We have managed to transform (1) into a non-homogeneous linear differential equation with constant coefficients. We can omit the constant ω, because the scale transformation $x\mapsto\omega x$ allows us to consider $\omega = 1$. Then, by integrating (2), we obtain the following sequence of equations:

$$z_{k+1}=\alpha\cos(x-x_a)-(\cos x)\int_{x_a}^{x} f(v,z_k,z'_k)\sin v\,dv+\beta\sin(x-x_a)+(\sin x)\int_{x_a}^{x} f(v,z_k,z'_k)\cos v\,dv. \tag{3}$$

Thus, we obtain a sequence of functions $S\equiv\{z_0(x),z_1(x),\ldots,z_n(x),\ldots\}$. We expect that $|z_{k+1}(x)-z_k(x)|\mapsto 0$ and $|z_k(x)-z(x)|\mapsto 0$ uniformly on $[x_a,x_b]$, where $z(x)$ is the exact solution of (1), see comments in Appendix A.

An alternative for the analytic study of the convergence is given by numerical tests. Then, in order to quantify the convergence of the sequence S, let us define the mean value of the square of the distance between two iterative steps on the interval $[x_a,x_b]$ as follows:

$$D(z_k,z_{k-1})=\frac{1}{x_b-x_a}\int_{x_a}^{x_b}(z_k(x)-z_{k-1}(x))^2dx. \tag{4}$$

In particular, the distance between a reference solution $u(x)$ and $z_k(x)$ is $D(u,z_k)$. We have to choose $z_0(x)$ in such a way that D as in (4) decreases as k grows. This can be done, since as can be seen from the methods given in [4–9] it is always possible to obtain an approximate solution. In particular, if we wish to obtain a periodic solution, it becomes quite convenient that $z_0(x)$ be periodic in order to speed up the convergence.

However, with the known methods, we have faced up to a big difficulty in some examples, since to an increase in the number of iterations it corresponds to an increase in the complexity of the non-homogeneous term $f(x,z_n,z'_n)$. Then, due to the limitations on the hardware, the search for the approximated iterated solution becomes extremely slow. We wish to stress that this inconvenient appears in widely used methods like the Lyapunov or the harmonic balance methods. Here is where our method presents some advantages over the traditional ones.

In the sequel, we shall discuss the precision of our method in several specific examples. The general Hill equation can be written in the form (1), although we shall consider here two particular cases only: the Mathieu equation and a fourth order Hill equation which appears in a physical problem. The second example here studied has to be with the van der Pol equation.

3. The Hill equation

As is well know, the Hill equation has the form $z''(x)+a(x)z(x)=0$, where $a(x)=\theta_0+\sum_{n=1}^{\infty}\theta_n\cos(2nx)$, where θ_n, $n = 0,1,2,\ldots$ are constants. If we write $a(x)=\omega^2-b(x)$, then the Hill equation adopts the form of our Eq. (1). It is well known that the Mathieu equation is a particular case of the Hill equation with $\theta_2=\theta_3=\cdots=0$. We can write the Mathieu equation as $z''+(\omega^2-2q\cos 2x)z(x)=0$. Since we are interested in periodical solutions, we choose $\omega = 2$ and $q = 1/4$. If we fix as initial values $z(0) = 1$ and $z'(0) = 0$, we obtain as solution $z(x) = C(2,1/4,x)$ where $C(\omega^2,q,x)$ is the cosine Mathieu function [18].

Then, starting with the test function $z_0(x)=\cos\sqrt{2}x$, we use the iterative formula given in Eq. (2) with $f(x,z_n) = 2qz_k\cos 2x$ and $\omega^2 = p$ in order to obtain the approximate (or iterate) solutions $\{z_1(x),z_2(x),\ldots,z_n(x),\ldots\}$.

Now, by using (4) for $x\in(0,10)$, we have that

k	0	1	2	3	4
$D(z,z_k)$	1.4×10^{-2}	1.8×10^{-3}	8.4×10^{-5}	5.7×10^{-6}	4.1×10^{-7},

while the distance between two consecutive approximate solutions is given by

k	1	2	3	4
$D(z_k,z_{k-1})$	1.6×10^{-2}	2.4×10^{-3}	1.1×10^{-4}	4.6×10^{-6}.

From this table of values, we can see that the iterate sequence of solutions converges to the exact solution on the interval $(0,10)$. The second approximate solution is given by

$$\begin{aligned} z_2 = {}&0.132027\cos 0.585786t+0.898438\cos 1.41421t-0.00804945\cos 2.58579t-0.0226523\cos 3.41421t\\ &+0.000236954\cos 5.41421t+0.0110485t\sin 1.41421t. \end{aligned}$$

If instead, we use the constant function $z_0(x)\equiv 1$ as test function, we obtain the following results: For the distance between the exact and approximate solutions, we find

k	0	1	2	3	4
$D(z,z_k)$	1.3	0.1	4.3×10^{-3}	4.2×10^{-4}	1.7×10^{-5}

and for the distance between two consecutive approximate solutions, we obtain